THE MATHEMATICAL GAZETTE

EDITED BY

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LONDON

G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2

Vol. XXII., No. 249. MAY, 1938. 5s. Net.

CONTENTS.

The Mathematical Association, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and is continuing to exert an important influence on methods of examination.

" The Mathematical Gazette " (published by Messrs. G. Bell & Sons, Ltd.) is the organ of the Association. It is issued at least five times a year. The price per copy (to non-members) is usually 3s. each.

The *Gazette* contains—Articles, Notes, Reviews, etc., dealing with elementary mathematics, and with mathematical topics of general interest.

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THE FEUERBACH QUADRILATERAL.*

BY N. M. GIBBINS.

NEARLY a month of the summer term remains to be got through by the somewhat jaded boys and girls after the Higher School examination is over. For many years I have been casting round for stimulating topics to discuss with my more or less advanced mathematicians, and this paper is concerned with my latest find. Its title refers to the four tangents at the points of contact of the nine-points circle of a triangle with the four circles touching the sides. I am aware that a great many properties of the points of contact themselves have been investigated, and I cannot flatter myself that I have found out anything new about the figure formed by the tangents thereat. But the literature of any branch of mathematics is so vast that it seemed best for me, instead of hunting in books, to try to find out what I could in my own way. The effort to do so resulted in a delightful mathematical adventure, which has enabled me to devise something to bring out to our pupils what analytical geometry really is about.

For indeed many boys and girls find it very hard to grasp the underlying principle of this branch of mathematics, and their attitude towards it is often like that of Jean Jacques Rousseau who says (*vide* the December *Gazette*): " I never got so far as to understand properly the application of algebra to geometry. I did not like this method of working without knowing what I was doing; and it appeared to me that solving a geometrical problem by means of equations was like playing a tune by simply turning the handle of a barrel organ." Rousseau has indeed stated the matter with precision, and I accept the challenge implied in this quotation. I propose to make a barrel organ, so to speak, and to grind out tunes from it. I venture further to hazard a guess that you will find the tunes to be good ones.

Once again leaving the shifting sands of metaphor for the security of an ordered exposition, what I shall do in this paper is to connect

* A paper read to the Annual Meeting of the Mathematical Association, 5th January, 1938.

the areal coordinates of a point with respect to the original triangle with its areal coordinates with respect to the diagonal triangle of the Feuerbach quadrilateral. Then by substitutions in well-known equations results ought to emerge. For instance, we can write down (or else look up) the equation of the join of the middle points of the diagonals of the quadrilateral when referred to its diagonal triangle; and then by a simple set of substitutions it turns out that this line is the Euler line of the original triangle. Or we can similarly write down the equation of the self-polar circle of the diagonal triangle, and by the same set of substitutions we find that it is the nine-points circle of the original triangle. I have purposely anticipated these results in order to show the kind of adventure I had, and I now proceed to illustrate the method of attack in a far simpler case by finding the equations of the four lines which are the subject of discussion.

1. In the figure, A', B', C' are the middle points of the sides of ABC. If x, y, z are the actual areal coordinates of a point P with

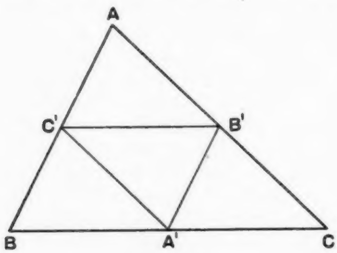

FIG. 1.

respect to ABC, and x', y', z' those of the same point with respect to $A'B'C'$, particles x, y, z at A, B, C respectively have a positive unit resultant at P, and particles $-x'$, $-y'$, $-z'$ at A', B', C' have a negative unit resultant at P. Thus the six particles x, y, z, $-x'$, $-y'$, $-z'$ are in equilibrium, and taking moments about $B'C'$, $C'A'$ $A'B'$ in turn we at once obtain the relations

$$x' = y + z - x, \quad y' = z + x - y, \quad z' = x + y - z.$$

If now (x', y', z') is a point on the circumcircle of $A'B'C'$, that is, on the nine-points circle of ABC, we have

$$\begin{aligned} 0 &= 4(a'^2y'z' + b'^2z'x' + c'^2x'y') \\ &= a^2\{x^2 - (y-z)^2\} + b^2\{y^2 - (z-x)^2\} + c^2\{z^2 - (x-y)^2\} \\ &= 2(a^2yz + b^2zx + c^2xy) - \Sigma(b^2 + c^2 - a^2)x^2 \\ &= 4\Sigma a^2yz - (x+y+z)\Sigma(b^2+c^2-a^2)x. \quad \text{......................(1.1)} \end{aligned}$$

The equation of the inscribed circle of ABC is

$$\Sigma(b+c-a)^2x^2 - 2\Sigma(c+a-b)(a+b-c)yz = 0, \quad \text{.........(1.2)}$$

or $$4\Sigma a^2yz-(x+y+z)\Sigma(b+c-a)^2x=0.$$

Hence the radical axis of the two circles is

$$\Sigma x\{(b^2+c^2-a^2)-(b+c-a)^2\}=0$$

or $$x/(b-c)+y/(c-a)+z/(a-b)=0. \qquad\text{.................(1.3)}$$

The equation of the ex-circle opposite A is obtained from (1.2) by simply changing the sign of a, while the right-hand side of (1.1) is the same function of $-a, b, c$ as it is of a, b, c. Hence the other sides of the Feuerbach quadrilateral of ABC are obtained by changing the signs of a, b, c in turn in (1.3).

2. The equations of the sides of the Feuerbach quadrilateral are therefore

$$-\frac{x}{b-c}-\frac{y}{c-a}-\frac{z}{a-b}=0, \qquad\text{......................(2.1)}$$

$$\frac{x}{b-c}+\frac{y}{c+a}-\frac{z}{a+b}=0, \qquad\text{......................(2.2)}$$

$$-\frac{x}{b+c}+\frac{y}{c-a}+\frac{z}{a+b}=0, \qquad\text{......................(2.3)}$$

$$\frac{x}{b+c}-\frac{y}{c+a}+\frac{z}{a-b}=0. \qquad\text{......................(2.4)}$$

By inspection the results of adding the first and second equations, and the third and fourth, differ only in sign. This is also true of the first and third, second and fourth equations; and of the first and fourth, second and third equations. The equations of the diagonals of the quadrilateral are therefore

$$\left.\begin{aligned}\frac{y}{c^2-a^2}+\frac{z}{a^2-b^2}&=0,\\ \frac{z}{a^2-b^2}+\frac{x}{b^2-c^2}&=0,\\ \frac{x}{b^2-c^2}+\frac{y}{c^2-a^2}&=0.\end{aligned}\right\} \qquad\text{......................(2.5)}$$

They pass through A, B, C respectively, and the new triangle $A'B'C'$ is in perspective with ABC, the equation of the axis of perspective being

$$\frac{x}{b^2-c^2}+\frac{y}{c^2-a^2}+\frac{z}{a^2-b^2}=0. \qquad\text{......................(2.6)}$$

The equation of AA' is $y/(c^2-a^2)=z/(a^2-b^2)$

or $(a^2-b^2)y+(a^2-c^2)z=0$, or $a^2(x+y+z)-(a^2x+b^2y+c^2z)=0$.

This is parallel to $a^2x+b^2y+c^2z=0$, that is, to

$$(a^2+b^2+c^2)(x+y+z)-2(a^2x+b^2y+c^2y)=0,$$

that is, to

$$(b^2+c^2-a^2)x+(c^2+a^2-b^2)y+(a^2+b^2-c^2)z=0, \quad \text{.........(2.7)}$$

which, from (1.1), is the radical axis of the circum-, nine-points, and self-polar circle of ABC. Similarly BB' and CC' are parallel thereto, and so therefore also is the join of the centroids G, G' of the triangles ABC, $A'B'C'$.

3. The preceding relation between the two triangles is a particular case of that between ABC and a triangle the equations of whose sides are

$$\frac{y}{\mu}+\frac{z}{\nu}=0, \quad \frac{z}{\nu}+\frac{x}{\lambda}=0, \quad \frac{x}{\lambda}+\frac{y}{\mu}=0,$$

where $\lambda+\mu+\nu=0$.

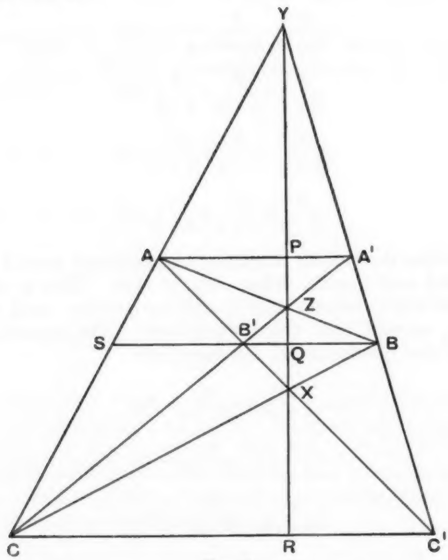

FIG. 2.

The following properties follow from Fig. 2, in which

$$A'B : BC' = \nu : \lambda.$$

The area $A'B'C'$ is half the area ABC. ....(3.1)

$$\lambda \,.\, AA' = \mu \,.\, BB' = \nu \,.\, CC'. \quad \text{......................(3.2)}$$

The axis of perspective XYZ cuts AA', BB', CC', GG' at their points of trisection nearer to A', B', C', G'. ....(3.3)

(For $SB'=B'B$ and $SQ/QB=CR/RC'=QB/B'Q$
$=(SQ+QB)/(QB+B'Q)=SB/B'B=2$; hence $QB=2B'Q$.)

Placing particles x, y, z at A, B, C, and particles $-x'$, $-y'$, $-z'$ at A', B', C' as in § 1, and taking moments about $B'C'$, $C'A'$, $A'B'$ in turn, the areal coordinates of a point P with respect to ABC and $A'B'C'$ are connected by the relations

$$x' = -\lambda\left(\frac{y}{\mu}+\frac{z}{\nu}\right),\ y' = -\mu\left(\frac{z}{\nu}+\frac{x}{\lambda}\right),\ z' = -\nu\left(\frac{x}{\lambda}+\frac{y}{\mu}\right); \qquad \text{......(3.4)}$$

so that
$$x'+y'+z' \equiv x+y+z. \qquad \text{........................(3.5)}$$

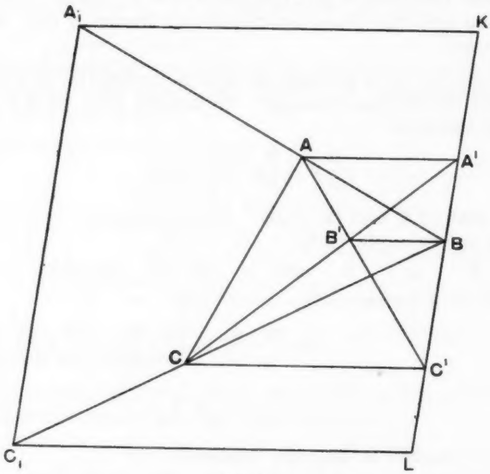

Fig. 3.

We obtain the sides of the triangle $A'B'C'$ from Fig. 3, in which $BK = BL = A'C'$, and BA and BC are produced to meet the parallels through K and L to AA' in A_1 and C_1 respectively.

We have $\quad A_1K/AA' = KB/A'B = A'C'/A'B = CC'/BB'$,

and $\quad C_1L/CC' = LB/C'B = C'A'/C'B = AA'/BB'$.

Hence $A_1K = C_1L$, and therefore $A_1C_1 = KL = 2A'C'$.

Also $\quad A_1B/AB = A_1K/AA' = CC'/BB' = \mu/\nu$, from (2.2);

and $\quad BC_1/BC = C_1L/CC' = AA'/BB' = \mu/\lambda$.

Hence
$$\frac{4b'^2}{\mu^2} = \frac{c^2}{\nu^2}+\frac{a^2}{\lambda^2}-\frac{2ca\cos B}{\nu\lambda}.$$

Hence
$$\frac{4b'^2}{-\lambda\mu\nu} = \left(\frac{1}{\nu}+\frac{1}{\lambda}\right)\left(\frac{c^2}{\nu^2}+\frac{a^2}{\lambda^2}\right)+\frac{\mu}{\nu\lambda}\cdot\frac{c^2+a^2-b^2}{\nu\lambda},$$

that is,
$$\frac{4b'^2}{-\lambda\mu\nu} = \frac{c^2}{\nu^3}+\frac{a^2}{\lambda^3}-\frac{a^2\lambda+b^2\mu+c^2\nu}{\nu^2\lambda^2}, \qquad \text{..................(3.6)}$$

with similar equations for a'^2 and c'^2.

In the present case $\lambda = b^2 - c^2$, $\mu = c^2 - a^2$, $\nu = a^2 - b^2$, so that $a^2\lambda + b^2\mu + c^2\nu = 0$, and equations (3.6) become

$$\frac{4a'^2}{-\lambda\mu\nu} = \frac{b^2}{\mu^3} + \frac{c^2}{\nu^3}, \quad \frac{4b'^2}{-\lambda\mu\nu} = \frac{c^2}{\nu^3} + \frac{a^2}{\lambda^3}, \quad \frac{4c'^2}{-\lambda\mu\nu} = \frac{a^2}{\lambda^3} + \frac{b^2}{\mu^3}, \quad \text{......(3.7)}$$

so that
$$\frac{2(b'^2 + c'^2 - a'^2)}{-\lambda\mu\nu} = \frac{a^2}{\lambda^3}, \text{ etc.} \quad \text{..................(3.8)}$$

It follows that one of the quantities cot A', cot B', cot C' must be negative and the other two positive. Hence $A'B'C'$ is always obtuse-angled.

4. We are now in a position to find the properties of the quadrilateral and its diagonal triangle. Equations (2.1) to (2.4) may be written in the form

$$\frac{a}{\lambda}x' \pm \frac{b}{\mu}y' \pm \frac{c}{\nu}z' = 0,$$

(forming with (3.4) and (3.7) the " barrel organ ").
Taking, for example, (2.4)

$$\frac{x}{b+c} - \frac{y}{c+a} + \frac{z}{a-b} = \frac{x(b-c)}{\lambda} - \frac{y(c-a)}{\mu} + \frac{z(a+b)}{\nu}$$
$$= a\left(\frac{y}{\mu} + \frac{z}{\nu}\right) + b\left(\frac{z}{\nu} + \frac{x}{\lambda}\right) - c\left(\frac{x}{\lambda} + \frac{y}{\mu}\right)$$
$$= -\frac{a}{\lambda}x' - \frac{b}{\mu}y' + \frac{c}{\nu}z'.$$

(We now proceed to play the " tunes ".)
When the equations of four lines are given in the form

$$px \pm qy \pm rz = 0,$$

the equation of the join of the middle points of the diagonals is

$$p^2x + q^2y + r^2z = 0, \quad \text{............................(4.1)}$$

and the radical axis of the circles on the diagonals as diameters is

$$\Sigma(q^2 - r^2)(b^2 + c^2 - a^2)(-x + y + z) = 0. \quad \text{..............(4.2)}$$

Here (4.1) becomes

$$\frac{a^2}{\lambda^2}x' + \frac{b^2}{\mu^2}y' + \frac{c^2}{\nu^2}z' = 0,$$

that is,
$$\frac{a^2}{\lambda}\left(\frac{y}{\mu} + \frac{z}{\nu}\right) + \frac{b^2}{\mu}\left(\frac{z}{\nu} + \frac{x}{\lambda}\right) + \frac{c^2}{\nu}\left(\frac{x}{\lambda} + \frac{y}{\mu}\right) = 0,$$

or
$$\Sigma\frac{x}{\lambda}\left(\frac{b^2}{\mu} + \frac{c^2}{\nu}\right) = 0,$$

that is,
$$\Sigma x\{b^2(a^2 - b^2) - c^2(a^2 - c^2)\} = 0$$

or
$$\Sigma x(b^2 - c^2)(b^2 + c^2 - a^2) = 0.$$

This is satisfied by $x=y=z=\frac{1}{3}$, and by

$$x=\cot B \cot C \equiv (c^2+a^2-b^2)(a^2+b^2-c^2)/16\Delta^2, \text{ etc.}$$

Hence it is the Euler line of ABC.

Equation (4.2) becomes, from (3.8),

$$\Sigma\left(\frac{b^2}{\mu^2}-\frac{c^2}{\nu^2}\right)\frac{a^2}{\lambda^3}(-x'+y'+z')=0,$$

which reduces to

$$3a^2b^2c^2(x+y+z)=16\Delta^2\,\Sigma(b^2+c^2-a^2)\,x,$$

and may be proved to bisect ON.

The most interesting property of the diagonal triangle is that its self-polar circle is the nine-points circle of the original triangle. We have, by (3.5) and (3.8),

$$\begin{aligned}
&2\Sigma x'^2(b'^2+c'^2-a'^2)/(-\lambda\mu\nu)\\
&\quad=\frac{a^2}{\lambda}\left(\frac{y}{\mu}+\frac{z}{\nu}\right)^2+\frac{b^2}{\mu}\left(\frac{z}{\nu}+\frac{x}{\lambda}\right)^2+\frac{c^2}{\nu}\left(\frac{x}{\lambda}+\frac{y}{\mu}\right)^2\\
&\quad=\Sigma\frac{x^2}{\lambda^2}\left(\frac{b^2}{\mu}+\frac{c^2}{\nu}\right)+\frac{2}{\lambda\mu\nu}(a^2yz+b^2zx+c^2xy)\\
&\quad=\{-\Sigma x^2(b^2+c^2-a^2)+2\Sigma a^2yz\}/\lambda\mu\nu.
\end{aligned}$$

Hence $2\Sigma x'^2(b'^2+c'^2-a'^2)=\Sigma x^2(b^2+c^2-a^2)-2\Sigma a^2yz,$ ....(4.3)
which proves the property.

Hence the orthocentre of $A'B'C'$ is the nine-points centre of ABC. The radii of the circles also are equal, that is,

$$-4R'^2\cos A'\cos B'\cos C'=\tfrac{1}{4}R^2$$

or

$$O'H'^2-R'^2=\tfrac{1}{2}R^2$$

or

$$NO'^2-R'^2=\tfrac{1}{2}R^2.$$

Hence the square of the tangent from N to the circumcircle of $A'B'C'$ is $\frac{1}{2}R^2$. Also since the self-polar circle of $A'B'C'$ is real, the latter triangle must be obtuse-angled.

We may also prove that the radical axes of the nine-points and self-polar circles of each triangle intersect on GK, where K is the symmedian point of ABC.

For

$$\begin{aligned}
2\Sigma x'(b'^2+c'^2-a'^2)/\lambda\mu\nu&=\Sigma\frac{a^2}{\lambda^2}\left(\frac{y}{\mu}+\frac{z}{\nu}\right)\\
&=\Sigma\frac{x}{\lambda}\left(\frac{b^2}{\mu^2}+\frac{c^2}{\nu^2}\right).
\end{aligned}$$

Now

$$\begin{aligned}
\frac{b^2}{\mu^2}+\frac{c^2}{\nu^2}&\equiv\left(\frac{b^2}{\mu}+\frac{c^2}{\nu}\right)\left(\frac{1}{\mu}+\frac{1}{\nu}\right)-\frac{b^2+c^2}{\mu\nu}\\
&=\left(\frac{a^2}{\lambda}+\frac{b^2}{\mu}+\frac{c^2}{\nu}\right)\left(-\frac{\lambda}{\mu\nu}\right)-\frac{b^2+c^2-a^2}{\mu\nu}.
\end{aligned}$$

Hence

$$2\Sigma x'(b'^2+c'^2-a'^2)+\Sigma x(b^2+c^2-a^2)$$
$$+\left(\frac{a^2}{\lambda}+\frac{b^2}{\mu}+\frac{c^2}{\nu}\right)(\lambda x+\mu y+\nu z)=0, \quad \text{......(4.4)}$$

where $\lambda x+\mu y+\nu z\equiv\Sigma x(b^2-c^2)=0$ is the join of (1, 1, 1) and (a^2, b^2, c^2). It is also the isotomic conjugate of the axis of perspective (2.6).

If the Euler line cuts $\Sigma x(b^2+c^2-a^2)=0$ in X and

$$\Sigma x'(b'^2+c'^2-a'^2)=0$$

in X' it may be verified that G is the point of trisection of XX' nearer to X. This, with the preceding property, gives a simple geometrical construction for $\Sigma x'(b'^2+c'^2-a'^2)=0$.

Again, from (4.3) and (4.4), it may be verified that

$$4\Sigma a'^2y'z'+2\Sigma x^2(b^2+c^2-a^2)$$
$$+\left(\frac{a^2}{\lambda}+\frac{b^2}{\mu}+\frac{c^2}{\nu}\right)(\lambda x+\mu y+\nu z)(x+y+z)\equiv 0.$$

Hence the radical axis of the circumcircle of $A'B'C'$ and the self-polar circle of ABC is GK.

It is probable that many more properties lie hidden in the original set of equations ; but I have shown enough of those I have brought to light by this method to make you realise that, next midsummer, I shall be able to share with my pupils something of the fun that can be got from " solving geometrical problems by means of equations ". N. M. G.

GLEANINGS FAR AND NEAR.

1179. In his spiritual experience Mr. Huxley has described the same curve which (although in a less wide perimeter) has been described by others of his generation. It is a curve which began in belief, traversed a long segment of disbelief, and has ended in belief again. More accurately, it is not a circle, but a spiral ; it began with dogma, it traversed a segment when the revolt against dogma implied a denial of any meaning, and it has ended in a conviction that some meaning either exists, or has got to be invented, which is beyond the limits of reason or intelligence. It is a belief, essentially, in the value of goodness.—*Daily Telegraph*, November 12, 1937. [Per Mr. D. Temple Roberts.]

1180. Had Gibbon followed his father's example and gone to Cambridge, he would have found the Mathematical Tripos fairly started on its beneficent career, and might have taken as good a place in it as Dr. Dodd had just done, a divine who is still, year after year, referred to in the University Calendar as the author of *Thoughts in Prison*, the circumstance that the thinker was later on taken from prison, and hung by the neck until he was dead, being no less wisely than kindly omitted from a publication, one of the objects of which is to inspire youth with confidence that the path of mathematics is the way to glory.—Augustine Birrell, " Edward Gibbon ", *Selected Essays*, p. 72.

CLARITY IS NOT ENOUGH.

By Lancelot Hogben, F.R.S.*

(An Address on the Needs and Difficulties of the Average Pupil.)

A striking thing about contemporary education is the increasing demand for mathematical training. In the eighteenth century the only departments of science which conspicuously called for it were astronomy and such branches of mechanics and optics as had arisen in connection with astronomical pursuits. Navigation was perhaps the only important profession for which any mathematical equipment was an indispensable prerequisite. Though an eighteenth century H. G. Wells might have anticipated minor avenues of future employment for professional mathematicians as teachers attached to artillery or actuarial work, he would scarcely have foreseen that chemistry, power production, genetics, psychology and social statistics would severally enlist the services of the mathematician. The educational problem which arises from the rapid mathematisation of science during the past half-century has found us unprepared, and is largely an unsolved one.

A generation has passed since Sylvanus Thompson created a storm in a teacup by claiming an intelligible introduction to the infinitesimal calculus as the birthright of the engineer. Thompson's crusade enlisted the support of the military and naval colleges. What he planted, Mercer watered. This was a substantial gain. We are now amazed at the ingenuity with which Edwards could have contrived to make his subject so repulsive to a healthy adolescent.

After Thompson came Mellor's plea for the student of physical chemistry. Partly because it relies largely on physical chemistry and partly in its genetical aspect as a science in its own right, biology is now facing the same issue. Elaborate analysis has been applied to the interpretation of experimental work on chromosome mapping, inbreeding and the relation of nature to nurture. The result of this is that experimental biologists often discover too late in life that theory has outstripped their early equipment. Here and there tentative efforts to start courses in biomathematics have been made in our own country and in America. In a few years the curricula of our schools and universities will have to take cognisance of the new demand.

Where conspicuous aptitude or partiality for mathematics exists, the task of the teacher calls for little enterprise, imagination or skill. So long as professional outlets for mathematical knowledge were restricted to a very limited range of professional interests, the supply of calculating prodigies was equal to the demand. While the demand was small enough to be satisfied by the available supply, there was no social incentive for studying the technique of mathematical exposition. Within the social class with access to a professional education, the boy who was naturally bright at mathe-

*Presidential Address to the London Branch of the Mathematical Association, 27th November, 1937.

matics could head for the navy—the others for the church, the bar or medicine. If a boy could not do mathematics he was deemed to be stupid, or to put it more charitably, his ambitions were canalised towards the episcopal benches in the House of Lords.

The present situation is that the demand for an irreducible minimum of mathematical proficiency vastly exceeds the supply of conspicuous natural talent. On that account the most signal contemporary contributions to educational technique are beginning to come from the ranks of mathematical teachers in technical institutes or military and naval colleges. Circumstances are forcing the teacher of mathematics to face a serious educational task, while his colleagues are occupied with topics which properly belong to the baby clinic, the consulting room or the juvenile court. The serious problems of education are not concerned with the " problem child ". They are concerned with the problem-subject. Mathematical teaching is the cardinal educational problem of our time, because mathematics is *par excellence* the problem-subject of the curriculum. The existence of the Mathematical Association is a healthy indication that some members of the teaching profession recognise the differences between pediatrics and pedagogy.

A presidential address may excusably mitigate the serious labours of the year by a seasoning of comic relief. So I shall choose my text from Stephen Leacock's essay on *How to Make Education Agreeable.* Those of you who have read it will recall Leacock's complaints against the prosaic language in which Euclid states that a perpendicular is made to fall on a line, bisecting it at a point C. Every competent journalist knows how to make this announcement sufficiently arresting by a judicious selection of type announcing in headlines of diminishing size :

AWFUL CATASTROPHE

PERPENDICULAR FALLS HEADLONG ON A LINE

Line at CINCINNATI completely CUT

President of the Line makes statement.

Whereafter the class, says Leacock, would be breathlessly eager to hear the President's statement.

Two Worlds of Discourse.

There is more in Leacock's little joke than a casual reading would suggest. Whatever else it is, mathematics is a technique of discourse for dealing with relations of size and order, in contradistinction to common discourse which is also (and more especially) concerned with relations of quality. *This limitation is inherent in the act of communication, and therefore asserts itself in the habits which a trained mathematician brings to the art of teaching.* From that fact the cardinal difficulties of mathematical teaching arise.

To be a good teacher and a good mathematician is almost as difficult as being a genuine saint and an expert politician. The world of discourse in which the mathematician lives is far from the untidy world of trial and error. Only the need for food at

irregular intervals forces him back to earth. His happier moments are spent in a better place. Orderly piles of related propositions are set out in neat rows along spacious avenues where accidents rarely happen. Even when they do, there is no sense of imminent tragedy. It is a world without people. Unlike the celestial city of St. John there are no children playing in the streets. When the need for food brings him into contact with real ones, the mathematician has long since forgotten the language in which they converse.

What seems to me the source of most difficulties which beset the average pupil can be best seen by contrasting the characteristics of mathematics with the technique of common discourse. In everyday life communication is an art which involves far more than exhibiting an orderly sequence of propositions. Common discourse is effective, and is endorsed as correct, in so far as it discharges two tasks simultaneously. One is to specify relations, or in less pompous language, to convey information. The other is to enlist the *personal* attention and engage the *personal* interest of the individual or individuals to whom the communication is made. The fact that a statement is instructive in the sense that it contains no ambiguity or error does not satisfy the requirements of ordinary communication. Conversely, mere brightness does not justify itself, if the content is shallow, equivocal or otherwise misleading.

That the informative and emotive aspects of communication are truly separate is recognised by grammarians who classify the artifices devised for harmonising their conflicting claims. There is an implicit recognition of this compromise in the definition of a *figure* of speech. The essentially personal relationship implied in the emotive function of ordinary language is recognised by the choice of form appropriate to the reading public or audience. No such distinction exists in the technique of discourse with which this Association is concerned. Brevity is the soul and substance of mathematical wit, and ambiguity or inconsistency are the only unpardonable sins. In short, there is no personal relationship involved in the act of communication.

In so far as mathematics has an aesthetic appeal, it is one which has no explicit relation to the history or prejudices of the individual as such. There are, to be sure, individuals for whom mathematics exerts a coldly impersonal attraction, and I suppose that the majority of professional mathematicians, the bright boys, belong to this category. This is not of great educational importance. Adolescents and children who are readily accessible to the austere aestheticism implied in the statement that a proof is elegant are rare. They are not the ones who make mathematics the problem subject of the classroom. Intense aesthetic satisfaction in mathematical pursuits is for most people an unattainable experience, or at the best an acquired taste. It is not, and cannot be, of itself a sufficient drive to proficiency. The teacher has to supply a powerful extrinsic motive of a more directly personal kind.

Many professional mathematicians show a sort of fussiness on this

topic, as though any statement of this kind necessarily casts doubts on the sincerity of their individual predilections. So in saying this let me forestall a criticism which will be made, whether I do so or not. The aesthetic appeal of mathematics may be very real for a chosen few. The point at issue is not whether it is real, but whether it is common. In fact, mathematics has little spontaneous appeal for the overwhelming majority of ordinary human beings. Otherwise there would be no special difficulties about teaching it. Of itself this makes teaching mathematics more difficult than teaching biology, because children are generally interested in their own bodies and in growing things.

Still this is not the aspect I propose to stress in what follows. The first and foremost difficulty lies less in the pupil than in the mathematician. The biologist, if he is a good biologist, brings to the teaching of his subject a recognition that a growing thing is something with its own laws of behaviour and that the business of teaching is closely allied to an understanding of these laws. The temperament and training of a mathematician do not encourage this outlook. Once he has been hardened to the rigours of the mathematical climate, he has ceased to be able to see simple truths which any journalist or mother of six takes for granted. Disraeli said that it is easier to be critical than to be correct. He might have added that most people are less anxious to be correct than to avoid criticism. Being correct in teaching is putting over as much of your meaning as you can. The fact that a scientific or mathematical exposition can be faultlessly clear to any one who can be induced to follow it is not of itself an educational achievement. If it is excruciatingly tiresome, no one will be induced to follow it. So its value as an act of communication will be nil.

Thirty or forty years ago a movement for the reform of mathematical teaching sponsored a luxuriant overgrowth of pedagogic literature with which we are still saddled. It amounted to this. Anyone with the wit to think up a slightly different arrangement of the propositions of Euclid could find a publisher and establish his credentials as an authentic pioneer. The educational world was shaken to its foundations by the daring announcement that Euclid had been banished from the class room, and the same dismal figures reappeared in the new text-books without any visible stigmata to distinguish them from illustrations in extant mediaeval translations of Al Karismi's works. In so far as this succession of tedious familiarities had any intelligent rationale, I suppose the theory behind it was to give the teaching of elementary mathematics a more coherent logical texture.

How far pioneers of pre-war days achieved their end is not of much interest in this context. My submission is that their aims were based on an assumption which is certainly gratuitous. The assumption is that mathematics is a problem-subject, because it makes exhorbitant demands for clear, consistent and closely-knit thinking. Whether this belief is right or wrong, it belongs to the

province of psychology rather than of mathematics. Great proficiency in mathematics confers no special qualifications for deciding whether it is true.

My own view is that it is entirely wrong. As a biologist I do not underestimate the prevalence of individual, often inborn, differences affecting the ease and quickness with which people can conduct comparatively complicated manipulations with symbols. On the other hand, it is obvious that the exercise of clear thinking is largely a matter of whether the individual is sufficiently interested in being right about a question, and has sufficient confidence in his own powers to face the strenuous effort involved in arriving at a correct conclusion. My contention is that mathematics is a problem-subject for two reasons. One is that no other subject offers so much temptation to be tedious. The other is that no other subject offers the teacher such great opportunities for wrecking the intellectual self-confidence of the pupil.

The first statement is illustrated sufficiently by the indisputable fact that the most atrocious examples of turgid, prolix and circumlocutory English style can be collected by opening at random the pages of any text-book on mathematics. That it is especially easy to discourage and to destroy interest in mathematics is a matter of common observation. Why so many grown-up people who give evidence of their power to handle comparatively difficult processes of reasoning in the course of their daily work display an attitude of sheer fright towards any mathematical formulation of a problem is easy to understand. At an early stage in education they have been made to feel that their difficulties were due to their own intellectual defects rather than to the cultural limitations of their teachers. The surest way of creating this sense of inferiority is to let children think that they are being taught for their intellectual improvement.

Thirty years ago I asked my class teacher why we had to learn geometry. I was told that it was a training for the mind. Twenty years later my eldest daughter asked her class teacher the same question and received the same answer. At different schools three of my four children have put the same question, unprompted by their parents. They have always received the same reply with penny-in-the-slot regularity. Since the odds against this result would be 7 to 1 if only 50 per cent., and 63 to 1 if only 25 per cent. of mathematical teachers took this attitude to their jobs, the result, though based on a small sample, is alarming. It is all the more alarming because the departure from random sampling favours the selection of good teachers. If we took education as seriously as Stalin takes engineering, we should punish the statement " it is obvious that " with the option of five pounds or thirty days.

When a mathematician ventilates his views on politics or biology you will rarely discover any confirmation for the belief that a native aptitude for clear thinking carries a man very far, unless it is reinforced by a genuine interest in what he is thinking about. If any first year medical student in my classes made the statements about

reflex action contained in Sir Arthur Eddington's *Gifford Lectures*, I should plough him. I should not do so because he did not know the relevant facts. I should do so because he had not learned to reason clearly about them, and I should do so knowing that I had failed as an educationist, in what is the supreme task of good teaching. Experience has taught me that there is a powerful psychological resistance to clear reasoning about reflex action. Perhaps this is because it touches our dignity on the raw. Be that as it may, it is my first duty to discover and to remove this resistance. The rest will look after itself.

In education the unforgivable sin is to attribute our own failures to the inferiority of our victims. The limitations of our pupils and students make education an exciting adventure, in which a certain kind of modesty is indispensable to success. Among the laity it is the general opinion that mathematicians regard themselves as exceptionally clever people. This may be quite unjust. None the less, it is a special reason for emphasising the cultivation of a modest demeanour as indispensable to the technique of mathematical teaching. Just as the training of a mathematician fails to stimulate the art of intimate and engaging discourse, it does little to encourage the sort of modesty which good teaching demands. Especially at Cambridge, where generalised ignorance about sex, politics, religion and most other topics of foremost importance to ordinary human beings is no obstacle to a brilliant career, academic success in mathematics makes peculiarly thrifty demands for a wide range of capacities. Since it calls for the exercise of a somewhat rare gift, mathematical students are especially liable to the delusion that quickness in manipulating intricate reasoning processes is all that distinguishes clever, capable and successful individuals from their less fortunate brethren.

Mathematics is a problem-subject primarily because the mathematician carries into the personal and individual relationship of teaching the impersonal detachment which properly belongs to a world of discourse in which symbols have no emotive value. If space permitted I should be prepared to sustain the view I have expressed by quoting large-scale statistical enquiries of educational psychologists. Here I will content myself by saying that few, if any, modern psychologists or biologists attach much meaning to the statement that any single discipline is specifically a training for the mind. What education can do is to encourage people to exercise to the full their capacity for intricate and rapid reasoning in relation to specific problems in which their interest has been quickened.

The So-What? *Difficulty.*

In my original notes for this lecture I wrote down the American idiom " so what? " to remind me of the difficulties which arise from failure to realise that *the art of being interesting is more important than the effort of being clear*. The class of difficulties which I include in the " so what " category may be illustrated by comparing the

teaching of chess with the teaching of determinants. An authentic chess player who wanted to convey his enthusiasm to someone who did not understand the game would begin his explanation by discovering whether his pupil or victim knew the rules of draughts or halma. He would indicate in a general way what the players aim at doing, and would explain what constitutes a win before facing the tedious and intricate task of memorising the rules, or studying the gambits and end games. If the same chess player adopted the educational technique employed in the chapter on determinants in any extant mathematical text-book, his explanation would begin somewhat as follows :

> A chess board is a board of eight rows and eight columns, among which are distributed two sets of sixteen pieces, eight alike of one kind being called pawns. The remaining eight include three pairs of which each member is alike of one kind, together with two other unlike pieces. The initial condition is that one of the pawns of one set occupy the second and the pawns of the other set occupy the seventh row. . . .

If the members of this audience took the same attitude to their work as those who taught me the elements of mathematics, I should enjoy retaliating for the tortures of the class-room by continuing the parable. In the present circumstances I think I have said enough to explain why I have called this the "so what?" difficulty. No mathematician would dream of expecting that anyone would pick up the rules of chess or acquire an enthusiasm for that noble game, if he got his first taste of it in this way. No reasonable person would feel compelled to attribute subnormal intelligence to anyone who resisted the attempt to be taught on these lines, or confessed complete failure to see what the teacher was driving at.

The primary task of the educationist is to establish the personal relationship of enlisting the personal interest of individual pupils in the exercise of their reasoning powers. Thus the problem of the mathematical teacher is not a problem of mathematics as such. The recipe for good mathematical teaching is to put into the teaching of mathematics something which does not belong to the subject-matter of mathematics as such. There are obviously many levels at which this can be done. As a biologist I cannot fail to notice that calf love for the teacher has sometimes supplied a powerful incentive to sustained effort. Happily or otherwise I can offer no general recipe for exploiting this technique, and I doubt whether an intensive teacher's course in Hollywood films would make it a measure of universal application. The aesthetic, which, at its most primitive level, is the play motive, is one which an enthusiastic and efficient teacher will not neglect, though I do not think it carries us very far by itself. To avoid any appearance of excessive utilitarianism in my subsequent remarks, let me make some tentative suggestions about how the play motive could be exploited more efficiently.

The first thing to be clear about in this connection is that there is not much fun in trying to solve a puzzle, unless you have some means of finding, or rather of convincing yourself that you have found, the right solution. From this point of view two types of elementary mathematics are obviously unsatisfactory materials for exploiting the aesthetic motive at its most primitive level. If a child is asked to solve a geometrical rider, the teacher's approval is his only way of finding out whether he has performed the prescribed ritual. That algebra is universally more popular than Euclidean geometry may be largely due to the fact that most problems arising out of school algebra can be checked by using ordinary numbers. On this account the solution of an algebraic problem contains an element of mildly exciting discovery. This is less true of permutations and combinations than of other elementary processes, because the labour involved in testing a result is usually prohibitive. The general practice of postponing permutations and combinations to a late stage may be taken as a tacit admission that they present difficulties out of all proportion to the logical processes involved.

In various ways much might be done to exploit the play motive, that is to say, getting children and adolescents to regard doing mathematics as real fun. One is perhaps more obvious to me as a biologist, because of the fruitful applications of Finite Differences to problems of selection and population. To the best of my knowledge no elementary courses touch on the type of series which can be illustrated by figurate numbers. Experience of children and the testimony of teachers who have carried out problems with figurate numbers at my suggestion have convinced me that they provide an almost inexhaustible fund of clean wholesome fun for children and adolescents who get no kick out of the customary, and, as Sir Percy Nunn has emphasised, unnecessarily retarded introduction to series by means of progressions.

Another mathematical device which is unnecessarily postponed till a comparatively late stage offers other intriguing possibilities for exploiting the play motive. Transformations from one numeral system to another can be made intelligible to very young children with the use of an abacus model, and the introduction of ancient systems like the Babylonian or Mayan calendrical numerals can give the treatment an engaging atmosphere of historical pageantry.

In passing, let me urge that we should resist the temptation to make the examination system an excuse for lack of enterprise in education. In the days when children were presented seriatim to third, second and first class college of preceptors examinations, the teacher had far less scope for departing from the routine prescribed by unimaginative text-books. That is no longer true. The junior examinations are disappearing. The secondary school teacher generally has a clear run of four years without any interruption.

I am sure that what I have found in teaching elementary biology to medical students applies *a fortiori* to the teaching of school mathematics. The fact that a syllabus is turgid, dull and uninspir-

ing makes it all the more important to stiffen the candidate for his ordeal by arousing his interest and enthusiasm. I believe that the teacher who scraps the conventional order in which school algebra and school geometry is presented and takes the trouble to stimulate his pupils by devoting a substantial part of his time to topics which lie quite outside the syllabus will get better examination results than the teacher who keeps one eye glued on the syllabus.

When I have had to teach within a syllabus prescribed by the General Medical Council I have usually devoted the first three-quarters of my course to the task of arousing the enthusiasm of my students for what I regard as real biology, *i.e.* what I enjoy teaching. That done, we can face up to the job of insuring ourselves against the fool questions which examiners ask in the remaining twenty-five per cent. of the allotted time. I have been eminently successful as a teacher of medical students by consistently carrying out this plan. So I am frankly cynical when school teachers put the blame for all their own shortcomings on the examination system.

If you have four years in which to get a class up to matriculation-level algebra, there is no earthly reason why you should follow the puerile routine of text-books in which identities and transformations, the algorithms, equations and progressions succeed with monotonous regularity. I believe that the teacher would attain his object in the time allotted with far greater success by spending a year playing with figurate numbers and numeral systems, deferring the introduction of any literal symbolism till it could be introduced to capitalise discoveries which any child of normal intelligence can make for itself.

The "so what?" problem in mathematics has two aspects. One already mentioned is the existence of a ritual for writing text-books to conceal the intrinsic interest of fresh technique. The other is the well-nigh universal absence of any attempt in text-books to enlist secondary drives on the part of the pupil by explaining the practical use to which the technique can be applied. The distinction I want to make can be illustrated by the opening paragraph of the chapter on Fourier's series in Gibson's *Elementary Treatise on the Calculus*. This begins with a bald announcement, which in common speech amounts to saying that some function of x may be represented by an infinite series of the sines and cosines of the integral products of x.

As Mr. Leacock would say, any smart pressman or columnist knows that this possibility has no news value when so stated. Its news value for the class-room depends on making quite clear *why it matters*. Until you have done this you have not tackled your first job as an educationist. You can do it in two ways: by exploiting its *intrinsic* and *extrinsic* interest. You can start by considering how predigested mathematical teaching leads you to have a hunch that such a series exists, and what class of mathematical puzzles could be solved by using its properties. You can also illustrate the human circumstances in which the problem arose by telling your class how

Fourier's original contribution was first used in connection with the study of heat conduction, and why the holy trinity of Laplace, Legendre and Lagrange refused to accept it for publication.

Humanising Mathematics.

My own views on exploiting the extrinsic or humane aspect of mathematics to enlist the interest of the pupil are sufficiently well known. On this topic I shall confine myself to three constructive comments.

One is to draw your attention to a curious anomaly. In exhibiting the *referability* of the methods expounded, new books at the *intermediate* stage, though written for a more selected group of pupils, take more pains than do elementary text-books for more general use. So while new books on the infinitesimal calculus are well illustrated with examples of its practical use in engineering or artillery, scarcely any widely-used text-book written for the matriculation course on geometry mentions its application to geographical truths which children are taught in the nursery.

An evident obstacle to educational progress in this direction is the exclusion of descriptive astronomy from the present curriculum. In none of the sciences are the relations of discovery to the social practice of mankind more clearly exhibited. Perhaps no other science is more relevant to information which most educated people have accepted on trust from their childhood onwards. Its neglect is all the more remarkable because of the close association of astronomy and navigation in the story of Britain's mercantile supremacy. One of the earliest things which most of us learned at school was that certain marks across a map were called lines of latitude, and that the world we live in is approximately twenty-five thousand miles in circumference. Although considerable time is devoted in schools to a subject called geography, most children still leave the secondary school, and one may venture to guess that most science graduates leave the university, without realising how a ship's captain determines the latitude of his vessel, and without hearing about the simple devices with which Eratosthenes measured the earth's boundary within fifty miles of the true value in the second century B.C.

Although a child of ten could find the latitude of his house correct to a degree on any clear night with the aid of a plumb line, a blackboard protractor and a couple of screws with eyes, most children take latitude, like the Copernican hypothesis, as an act of religious faith, and if they are Protestants, think it odd that Catholics refused to accept it in the same spirit. The infusion of a little more elementary astronomy into the teaching of geography would raise one of the dullest school subjects to the dignity of a rational discipline, and incidentally revolutionise the teaching of elementary mathematics by providing illustrative materials of the class of problems with which the more elementary branches of mathematics were designed to deal. The new departments of geography in the universities could make a welcome and fundamental contribution to the equipment of a social personnel competent to advance the

cultural aims of science, if they made a course in the methods and history of cosmography and of calendrical practice compulsory for their own students and optional for students in natural sciences. Professor E. R. G. Taylor of Birkbeck College is to be congratulated on her initiative in this matter.

Another welcome innovation has taken place at University College, where Professor Woolf has offered a course on the history of science and technology for students of education. Courses of this kind in the Departments of Education of our universities and in our training colleges could provide a focus for cultural collaboration between the historian and the man of science. This suggestion must needs run the gauntlet of a powerful, and at the same time pardonable, body of prejudice, which is expressed in a recent circular of the Board of Education. It has arisen because of a fashion which was once adopted to enliven the teaching of some sciences, notably chemistry and physiology, in the universities. Later on it percolated into school text-books. It was called the historical approach, because the tedium of the lecture room was from time to time relieved by lantern slides of bearded and very much superannuated scientists or of their birthplaces. Many of us can still recall how funeral cards of great-uncles who have gone before have helped us to return to the matter in hand with redoubled zest.

No doubt this method of instruction had the merit of familiarising students, who would not read Dr. Pflugel's works, with sartorial styles of earlier periods. As it affected a general outlook, it left the impression that science had progressed by a succession of miraculous divinations of exceptionally gifted individuals, who might have contrived to be born at any convenient time with much the same results. Needless to say biographical anecdotage of this sort does not serve any useful purpose which an historical approach can fulfill. When the history of mathematics is taught as a record of the progress of intellectual achievement it can do two things. By exhibiting the social background of mathematics and its uses in the common life of mankind, it invests the study of mathematics with human value for those who are indifferent to its own austere beauty. By emphasising its gradual growth it assuages and restores the self-confidence shattered by repeated assertions that a statement is *obvious*. For ordinary mortals it is reassuring to find that a supposedly obvious statement has defied the collective effort of all the best intelligences for several centuries.

The Prevalence of Mathematical Amnesia.

Another psychological error which arises from undue preoccupation with the logical difficulties of mathematics is that too little attention is paid to the part which memory plays in performing intricate mathematical operations. Teachers do not always remember to remind their pupils that certain things must be thoroughly *memorised*. The teacher's anxiety to gain the rational assent of his pupil often leads him to forget that memorising the result is just as

essential in mathematics as in any other subject. Solving a partial differential equation by Fourier's series is just as much a feat of memory as correctly describing the characteristics of a flowering plant when you only know its generic name. The teacher who is never tired of urging his pupils to work out their problems from first principles should be deprived of his next meal till he has repeated Kelvin's unhappy calculation on the age of the earth without assuming anything.

What is easily overlooked in relation to this aspect of mathematical teaching is that memory is exceedingly capricious. Facility in remembering different types of information is highly individualised. Speaking for myself, I may illustrate this by the fact that, although I like organic chemistry, I can carry few facts about carbon compounds in my head for more than a few weeks. Contrariwise I have an encyclopaedic memory for trivial anatomical facts, which I have had no occasion to recall for twenty-five years. Among my students I have often had individuals who shone at mathematics and found the greatest difficulty in remembering simple zoological terms when their derivation and significance had been repeatedly explained.

As a teacher of biology I recognise this as a special class of difficulties, and I strive to resist the temptation to blame the student for lack of ability to reason correctly about living creatures and their characteristics. One real difficulty which makes some pupils slow is not lack of ability to follow mathematical reasoning. More often than we care to admit, it may be failure to capitalise results which must be committed to memory before complicated operations can be performed with alacrity. In all teaching it is advantageous to set out periodically a summary of information already acquired for careful memorisation. This provides a platform for the next stage in the development of the subject, and makes it easier to trace successive stages in retrospect.

No doubt our Victorian grandparents laid too much stress on the role of memory in education. None the less the pendulum has now swung too far in the opposite direction. It is high time to tell Aldermen, Rotary presidents and Gifford lecturers that they do not justify their claims to originality in educational theory by warning us against cramming our children with facts. When we have done our best to interest, to stimulate, to win the confidence and to gain the rational assent of those we teach, they have got to do some real work themselves. The major part of it is systematic memorisation of what they should have once understood if we have done our job well. Part of doing it well is also to encourage them to undertake systematic memorisation at each stage.

What I have said so far has been especially about pure mathematics, and may be summed up by saying that the teacher's job is less to make things clear than to give his pupils a powerful incentive for getting things clear for themselves. Three obstacles which he has to surmount are easy to recognise. The first is the paralysing

sense of *unfamiliarity*, which I have called the "so what?" reaction. The second is a sense of *intellectual inferiority*, which discourages further effort. The third is a disinclination for studies with no explicit *practical outcome*. This is common among healthy extroverts, even those who have a high level of intellectual capability. The teacher can forestall these difficulties, first by intelligent *anticipation*—the "so what?" technique—by cultivating a *modest friendliness* of deportment and by equipping himself with the information to give his treatment of any problem the widest possible *referability*.

The Black Sheep Difficulty.

The last remark might be taken to imply that the average pupil encounters less difficulties in studying applied than pure mathematics. Few teachers would agree that this is so. Many adolescents with no special partiality for mathematics prefer the pure to the applied sort, and experience more difficulty with the latter. I do not believe that they belong exclusively to the introvert type. On the contrary, my opinion, for what it is worth, is that the extrovert resistance to applied is just as great and often greater than to pure mathematics.

Where this resistance exists, one difficulty of the average pupil is easy to recognise. In applied mathematics the pupil has to deal simultaneously with two intellectual problems. One is the ordeal of performing certain operations with symbols. The other is the relation between the counters and the real world. The latter is often made more difficult by the fact that the symbols are not the current coin of the realm of nature itself. They refer to the characteristics of a physical model, and not to the natural processes which are the ostensible topic of discussion. Inability to perform the prescribed ritual may, and I believe often does, result from preoccupation with the relevance of the symbols to the model or of the model to the process itself.

That this is not a wild surmise is easy to illustrate by observing the reactions of different individuals to problems of the type contained in the Week-end Book. The essential ingredient of any Bloomsbury conundrum is some irrelevant circumstance or unfamiliar situation which distracts attention from what would otherwise be a straightforward issue. At the lowest level of naïveté this may be illustrated by the familiar riddle: do white sheep eat more than black ones? Everyone knows that individual white sheep do not eat conspicuously more than individual black ones, and that the latter are less common. The difficulty is easy to see. The *milieu* suggests that the right answer demands biological knowledge rather than arithmetical commonsense.

Like many others among my own contemporaries I probably got my first taste—or distaste—for mechanics from the same source as many members of this audience. The prescribed ritual of school mechanics has not moved very far in my own generation. The questions set in London University examinations, at all stages from

matriculation to the degree, conduct the successful candidate through a labyrinth of tedious and devious algebraic manipulations to conclusions whose falsity is self-evident to any schoolboy with a taste for gadgeteering or to any practising mechanic.

The black sheep problem in the wolf's clothing of academic mechanics is illustrated by the following example taken from Loney's treatise on the *Dynamics of a Particle* :

> "Assuming that the earth attracts points inside it with a force which varies as the distance from its centre, shew that, if a straight frictionless airless tunnel be made from one point of the earth's surface to any other point, a train would traverse the tunnel in slightly less than three-quarters of an hour."

By the time the student has reached the degree stage oral tradition among undergraduates has equipped him with enough low cunning to detect the trick. He realises that all the examiner requires is a piece of plain painstaking arithmetic according to a prescribed pattern. The engineering, like the genetics in the black sheep problem, is merely put in to make it more difficult. At the school certificate stage there is no robust corpus of undergraduate tradition to inoculate the pupil against the unnecessary distractions of our educational routine. So the teacher's task demands more intelligence.

The principles of mechanics discovered by Stevinus, Galileo, Hooke, Huyghens and Newton have very little relevance to the mechanisms with which a boy is familiar. In real life he never meets perfectly smooth bodies sliding down perfectly flat slopes without any friction. He has more experience of motor bicycles which seize because of overheating. If he were enterprising enough to calculate the trajectory of Big Bertha when it shelled Paris, he would find that the actual range and height were less than half what would be inferred from the formula given in the text-book. Of course, few boys would be so enterprising. The boy who was would have the making of a scientist in him, and the best way of training a scientist is not to start him off with wrong ideas about the way the world works.

As long as the teacher has to prepare pupils to pass examinations in mechanics conducted in the usual way, he will find his task easier if he tells the whole truth. Half the truth, of course, is that the principles of mechanics in the Newtonian epoch were not designed to deal with modern mechanisms. So we must not be surprised or disappointed if they have to be supplemented by much more information before they can give us a useful guide to conduct in the everyday life of a secondary school pupil who lives in the age of the light car and the autogyro. A conscientious teacher will generally point this out, and leave the pupil wondering why it is necessary to learn the principles if they do not fit the facts. So the other half of the truth, more rarely disclosed, is equally important. Galilean mechanics did provide a very useful guide to conduct in an age when sailing ships were first undertaking westerly courses to uncharted oceans.

The Difficulty about Approximations.

In contradistinction to the " so what ? " problem of allaying the sense of unfamiliarity or futility which discourages effort in pure mathematics, the *black sheep* problem of realism in applied mathematics may be discussed at various levels of relevance and at different levels of sophistication. At the lowest we may recall examples in compound proportion concocted to illustrate the untruth that too many cooks never spoil the broth. At a later stage we should distinguish between two different ways of applying mathematics to the real world. Galileo's trajectory illustrates one, Maxwell's hypothesis the other.

Galileo's trajectory is a synthesis of quantitative laws which approximately describe the behaviour of Galilean cannon balls in certain specified conditions, one being the principle of inertia employed in rectilinear markmanship, and the other being the constant initial acceleration of heavy bodies falling to earth. To be quite clear about what it involves, it is necessary to specify the conditions, the limits of observational error, and the goodness of fit.

Generally, as with Galileo's pendulum, an additional source of psychological irrelevance is introduced into theories of this class by making approximations in the mathematical synthesis itself. For instance, the statement that the period of a simple pendulum swinging in a small circular arc is approximately constant implies more than the fact that it is liable to sources of error involved in the law of the inclined plane or in any other Galilean theorem from which you care to derive it. When you make use of the limit $\sin A = A$ radians, you have made the pupil's foothold in the real world less secure, unless you have taken the initial precaution of tabulating the numerical errors involved for different angles of swing.*

In the elementary example cited the approximation is in the last stages of the development. In more advanced problems of applied mathematics it is often buried in the brickwork of a formidable architectural feat. Approximations are made to transform expressions which could not otherwise be reduced by means of familiar artifices. At this level it is almost impossible to determine the limits of validity involved, and to do so would require elaborate analysis. The bewildered student taken tortuously through the formalism of the earlier steps, now feels that he has been led up the garden path. Sometimes he *has* been led up the garden path. His criticism is mathematically valid. Educationally, this is the nemesis of a spurious rigidity. If applied mathematics were treated as the tool it is, and not as an end in itself, the student would look *forward* to the end-product of the development as the *means* of testing its truth. Instead, he feels as if he has been caught in the act of cooking his result.

* There is a further source of psychological resistance to this. Young students and pupils are used to the *degree* as the unit of angular measurement, and the first reaction to a " small angle " is to identify it with something less than one degree. It is very easy for a beginner to forget that even 5° is a relatively small fraction of a radian.

Whenever a pupil is told that something may be neglected because it is small, his attention is immediately diverted from the logical texture of the interpretation. He finds himself asking how small it may be before it is entitled to mathematical exemption. You cannot sustain his confidence or enthusiasm, if you leave him spluttering with the uncertainties of cosmic untidiness in mid-stream. A teacher of applied mathematics should teach his subject as he would teach swimming, if he enjoyed it. The average pupil is like a young swimmer, who can just keep afloat. He knows the strokes and can execute them with tolerable proficiency, so long as he concentrates his effort on keeping his head above water. Approximations are like the first wave which sends water up his nostrils. He gulps, abandons hope and clutches at anything within reach. If you do not want him to sink, and do want him to learn to swim, you have got to teach him to breathe. Half the difficulty of learning to swim is learning to keep your grip on reality by controlling the act of breathing.

This homely truth is sometimes dismissed by asserting that mathematical teaching is concerned with the logical structure of scientific hypothesis without regard to its truth. I do not believe that scientific truth and scientific logic can be kept so far apart. A scientific law is not correctly stated, unless it contains within itself a recognition of its own limitations. In so far as applied mathematics is part of the methodological background of science, the mathematician who fails to clarify the sources and limits of numerical approximation is not fulfilling his role as logician.

Models and Metaphors.

In contradistinction to most problems of school mechanics *Ether* theories illustrate a higher level of sophistication. The quantitative laws which are found to give a good fit to observed data have not been derived by direct observation of processes with which theory deals. They have been drawn from behaviour of physical models first suggested by crude and superficial similarities.

The edifice of scientific knowledge is supported by a scaffolding of deceased and dying metaphors. In the boyhood of science the props of the building were green shoots still rooted in the soil of daily experience, and the giants of physics and chemistry played hide and seek among them. Analogies drawn from familiar experience of sailing boats or watermills, and mathematical operations suggested by the same analogies, once furnished fruitful clues for research. The foundations of mathematical theories of immense importance in science were laid by men who could visualise these analogies vividly.

We too easily forget that the experience of everyday life in the eighteenth century is not our own, and that parables drawn from it have ceased to be vivid. When modern text-books of physics compare the electric current with the flow of water to illustrate the characteristics of the former, they inevitably fail to achieve the end in

view. They merely tell pupils who have already picked up something about electricity from everyday life in the nineteenth century something which eighteenth century schoolchildren knew about very elementary hydrodynamics.

In dealing with this difficulty of the average student the teacher of mathematics may turn to the Church for guidance. The parable of the mustard seed does not contain any doctrinal truths which defy the comprehension of the most obstinate unbeliever. Any curate who knows his job recognises that the real difficulty about the mustard seed parable is connected with the conventions of systematic botany or with the physiology of plant growth in an oriental climate. The serious business of preparing his sermon is to straighten out this tangle, and he takes you back to Palestine accordingly. The serious business of the teacher who wants to help the average student of electricity to understand Maxwell's equations is to make him visualise Maxwell's vortices and Maxwell's cosmic half-set jelly as vividly as Maxwell himself envisaged them.

Needless to say, vivid imagery which would enlist the imagination of the pupil or student in the intellectual task he is undertaking is the very last thing which will be found in text-books of mathematics. By the nature of his training the mathematician brings to the lecture room, to the class room and to the making of a text-book a fastidiousness which is the father of tedium. Determined at all costs to avoid the pitfalls of ambiguity, he reduces the art of discourse to a lifeless jargon in circumstances where no danger of misunderstanding exists. The good English word *something* is not sufficiently technical. The applied-mathematician is only happy when he has called it a *body*, and has thus made the sentence in which it is buried a coffin.

My last remarks embrace all that I have tried to say in this lecture. Mathematics, as I see it, is primarily a problem-subject, because teaching mathematics successfully is very much like teaching anything else successfully. The first business of the teacher, whatever his subject, is to make it interesting, to discover what discourages his pupils, and to convey to them some of his own enthusiasm.

Some Overdue Reforms.

I believe that much could be done to improve the teaching of mathematics without any radical change in the examination system or in the general policy of education. For that reason I have tried to avoid any reference to innovations which would make the teacher's task more easy. To meet the new demands for mathematical proficiency, I venture to suggest that the most urgent syllabus reform is a wholesale reduction of formal plane geometry to make way for a much earlier introduction to trigonometry, analytic geometry, calculus and solid figures. I also suggest that this might be helpfully supplemented by closer co-operation with the teaching of geography and elementary descriptive astronomy in the schools.

As it seems to me, the need for university reform is equally

urgent. Mathematical graduates often leave the university completely ignorant of the history of their subject, and do not invariably possess any acquaintance with ordinary mathematical appliances such as almanacs, slide rules, verniers, or even computing machines. The aim is to develop an almost pathological proficiency in a limited range of technique, often of a kind which has long since outlived its usefulness or has not yet been proved to have any.

Beyond this looms the sinister fact that the task of teaching mathematics in Britain is hampered by an antiquated system of weights and measures. This is disastrous for education in more than one way. One is that the child is forced to undertake tedious arithmetical exploits at an age when he has no interest in manipulating large numbers. Hence he has often acquired an active hostility to number-lore long before he takes up the systematic study of mathematics. Another is that the current coin of physical discourse has little relation to the units of the practising engineer. This makes exact science a cult which has no very obvious relation to everyday life in Britain. In France the introduction of the metric system was only gained at the cost of a revolution. Mr. A. P. Herbert has shown that a university member can compel our own Parliament to make divorce easier. Is it too much to ask mathematical teachers to insist that every parliamentary representative for a university seat should give an undertaking to introduce a bill for the remarriage of arithmetic with commonsense?

Mathematicians have another political responsibility connected with arithmetic. The recent analysis undertaken by Dr. Enid Charles shows that whatever changes in fertility and mortality may conceivably conspire to arrest a rapid decline of net population from 1945 onwards, nothing can now forestall a rapid and spectacular depletion of the school age groups during the *next two decades*. Therefore the choice lies between a period of acute unemployment for teachers or a drastic reform of educational routine. No teacher should teach for more than ten hours a week. By 1950 an enormous reduction of working hours can be achieved without any increase in the cost of education. My last word is to suggest that teachers of mathematics should take the lead in seizing this opportunity. They will find the facts in a new book, *Political Arithmetic*,* which contains Dr. Charles' estimates of the oncoming shift in the school population.†

* Published by Allen and Unwin.

† I sent this in proof to Professor Levy, to whose friendship I owe more intellectual stimulus than I can ever repay. He calls my attention to a serious omission, and I cannot resist the temptation to quote his words:

"There is too much teaching of the pupils by the teacher. It is possible so to arrange matters that the pupils can teach the teacher. As you know, I have since I began teaching always encouraged my class to bring stuff to me to tackle in front of the class without any previous preparation on my part, so that the students can see how I flounder about until I get to the answer, or can see me being stumped. We then make a joint study of why I floundered or why I was stumped. They enjoy that, and they learn a tremendous amount in the process."

ANOTHER ETERNAL TRIANGLE.

BY N. M. GIBBINS.

IN the textbooks there are many examples concerning the leading features of the triangle whose sides are given by the equations $ax^2+2hxy+by^2=0$ and $L \equiv lx+my+n=0$. These are usually easy because of the use that can be made of the origin. But when we replace the first equation by the general equation

$$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

the matter is not so simple, and it is the object of this article to exhibit a method which gets over the difficulties. The "rule of the game" is that S must never be resolved into factors. Instead, use is made of pure geometry, and the formula for the power of a point with respect to a circle. Moreover, plenty of practice is provided in the use of abridged notation.

1. *Mensuration of the triangle.*

We have $$\tan A = 2(h^2-ab)^{\frac{1}{2}}/(a+b), \quad \text{.......................(1)}$$

and we require a formula connecting the angles.

In the figure AB, AC are $S=0$; BC is $L=0$; P is a point (x', y') on BC, and PFE, PGD are perpendicular to AC, AB respectively.

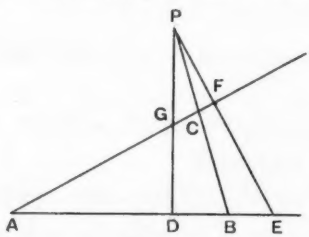

FIG. 1.

We have $$PD = PB \sin B,$$

and $$PF = PC \sin C = PG \cos A.$$

Hence $$\frac{PD \,.\, PG}{PB \,.\, PC} = \frac{\sin B \sin C}{\cos A}.$$

Now for the circle $DEFG$,

$$ax^2+2hxy+by^2+\ldots+b(x-x')^2-2h(x-x')(y-y')+a(y-y')^2$$
$$\equiv (a+b)(x^2+y^2)+\ldots=0,$$

whence $$PD \,.\, PG = S'/(a+b); \quad \text{.........................(2)}$$

and for the circle ABC,

$$ax^2+2hxy+by^2+\ldots+(lx+my+n)(\lambda x+\mu y+\nu)$$
$$\equiv \kappa(x^2+y^2)+\ldots=0,$$

so that $lx+my$ must be a factor of

$$ax^2+2hxy+by^2-\kappa(x^2+y^2).$$

Hence $$\kappa=(bl^2-2hlm+am^2)/(l^2+m^2), \qquad (3)$$

and $$PB\,.\,PC=S'/\kappa, \qquad (4)$$

since $L'\equiv 0$.

Hence $$\frac{bl^2-2hlm+am^2}{(a+b)(l^2+m^2)}=\frac{\sin B\sin C}{\cos A} \qquad (5)$$

Equations (1) and (5) determine the angles of the triangle.

Let the coordinates of A be x_0, y_0, so that

$$x_0=(hf-bg)/(ab-h^2), \quad y_0=(gh-af)(ab-h^2).$$

Then $$L_0=\begin{vmatrix} l & m & n \\ a & h & g \\ h & b & f \end{vmatrix}\div(ab-h^2).$$

We can now calculate other elements of the triangle. We have, for example,

$$2R\sin B\sin C=\text{the altitude through } A$$
$$=L_0/\surd(l^2+m^2).$$

Hence, from (5),

$$2R\cos A=L_0(a+b)(l^2+m^2)^{\frac{1}{2}}/(bl^2-2hlm+am^2),$$

and, from (1),

$$BC=2R\sin A=2L_0(h^2-ab)^{\frac{1}{2}}(l^2+m^2)^{\frac{1}{2}}/(bl^2-2hlm+am^2).$$

Hence the area of the triangle

$$=L_0{}^2(h^2-ab)^{\frac{1}{2}}/(bl^2-2hlm+am^2).$$

The square of the radius of the self-polar circle is

$$-4R^2\cos A\cos B\cos C=4R^2\cos^2 A\left(-\frac{\cos B\cos C}{\cos A}\right)$$
$$=\frac{L_0{}^2(a+b)(al^2+2hlm+bm^2)}{(bl^2-2hlm+am^2)^2},$$

on subtracting each side of (5) from unity.

Thus the condition that ABC should have an obtuse angle or be acute-angled is $(a+b)(al^2+2hlm+bm^2)$ positive or negative.

If either of the angles B or C is equal to A, the right-hand side of (5) becomes $2\sin^2 A$; whence, again subtracting each side of (5) from unity, the condition that $L=0$ may lie along one of the equal sides of an isosceles triangle made with $S=0$ is

$$(al^2+2hlm+bm^2)/(a+b)(l^2+m^2)=\cos 2A$$
$$=(1-\tan^2 A)/(1+\tan^2 A).$$

2. *The altitudes and the orthocentre.*

We now require the result that if $S \equiv f(x, y)$, then

$$f(x+\lambda r, y+\mu r) \equiv S + 2r(\lambda X + \mu Y) + r^2(a\lambda^2 + 2h\lambda\mu + b\mu^2), \quad \text{.....(6)}$$

where $X = ax + hy + g, \quad Y = hx + by + f.$

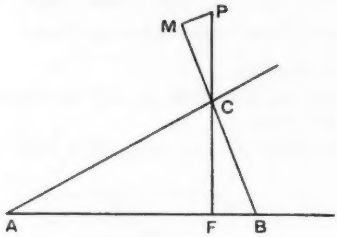

Fig. 2.

Let P be a point on the altitude CF. Draw PM perpendicular to BC. Then P, M, F, B are concyclic,

so that $$PC \,.\, CF = MC \,.\, CB,$$

or $$PC(PF - PC) = MC(MB - MC),$$

or $$PC \,.\, PF = MC \,.\, MB + PM^2.$$

If P is (x, y) and M is (x', y'),

$$x' = x - \frac{lL}{l^2+m^2}, \quad y' = y - \frac{mL}{l^2+m^2}.$$

Hence, by (6),

$$S' = S - \frac{2L}{l^2+m^2}(lX + mY) + \frac{L^2}{(l^2+m^2)^2}(al^2 + 2hlm + bm^2),$$

while $$MC \,.\, MB = S'/\kappa \text{ and } PC \,.\, PF = S/(a+b).$$

Also $\kappa \,.\, MC \,.\, MB + \kappa \,.\, PM^2$

$$= S - \frac{2L(lX+mY)}{l^2+m^2} + \frac{L^2(al^2+2hlm+bm^2)}{(l^2+m^2)^2}$$
$$+ \frac{(bl^2 - 2hlm + am^2)}{l^2+m^2} \cdot \frac{L}{l^2+m^2}$$
$$= S - \frac{2L(lX+mY)}{l^2+m^2} + \frac{L^2(a+b)}{l^2+m^2}. \quad \text{.............................(7)}$$

Hence the equation of the altitudes through B and C is

$$\frac{S}{a+b} \cdot \frac{(bl^2 - 2hlm + am^2)}{l^2+m^2} = S - \frac{2L(lX+mY)}{l^2+m^2} + \frac{L^2(a+b)}{l^2+m^2},$$

or $$(al^2 + 2hlm + bm^2)S - 2(a+b)L(lX+mY) + (a+b)^2L^2 = 0.$$

The equation of the altitude through A is

$$(x-x_0)/l=(y-y_0)/m\equiv\lambda, \text{ say ;}$$

substituting $x=x_0+l\lambda,\ y=y_0+m\lambda$ in the above, we obtain

$$[(al^2+2hlm+bm^2)\lambda-(a+b)\{L_0+\lambda(l^2+m^2)\}]^2=0.$$

Hence $$\lambda=-(a+b)L_0/(bl^2-2hlm+am^2),$$

and the coordinates of the orthocentre are found.

3. *The circles of the triangle.*

(i) If P is a point on the circle on BC as diameter, and PM is perpendicular to BC, then

$$PM^2=BM\,.\,MC \text{ or } MB\,.\,BC+PM^2=0.$$

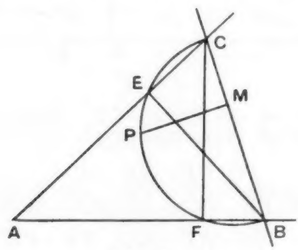

FIG. 3.

Hence from (7), the equation of this circle is

$$(l^2+m^2)S-2L(lX+mY)+(a+b)L^2=0, \quad \text{.............(8)}$$

in which the coefficient of (x^2+y^2) is $bl^2-2hlm+am^2$. This is the circle from which the other circles are obtained, just as the circumcircle is the basic circle in trilinear coordinates.

(ii) The circumcircle of ABC is that member of the coaxal system,

$$(l^2+m^2)S-2L(lX+mY)+(a+b)L^2=\lambda L$$

which passes through $A\,(x_0, y_0)$.

Since $$S_0=X_0=Y_0=0,$$
$$\lambda=(a+b)L_0,$$

and the equation becomes

$$(l^2+m^2)S-2L(lX+mY)+(a+b)L(L-L_0)=0. \quad \text{.........(9)}$$

Putting $S=0$ in this, the equation of the tangent at A is

$$(a+b)(L-L_0)-2(lX+mY)=0.$$

If P is a point on the tangent at B or C (Fig. 4) and PM is perpendicular to BC, $PB=PM$ cosec A ; whence, squaring and using the power of P,

$$\{(l^2+m^2)S-2L(lX+mY)+(a+b)L(L-L_0)\}/(bl^2-2hlm+am^2)$$
$$=(L^2\operatorname{cosec}^2A)/(l^2+m^2),$$

which is the combined equation of the tangents at B and C. Putting $S=0$ in this, the equation of the join of the points in which the tangents at B and C meet the opposite side is

$$(a+b)(L-L_0)-2(lX+mY)=\kappa\,.\,\mathrm{cosec}^2 A\,.\,L,$$

which passes through the meet of BC with the tangent at A.

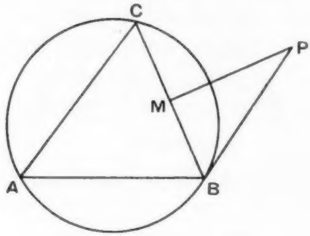

FIG. 4.

(iii) In Fig. 3, the equation of EF is, from (8),

$$M\equiv(a+b)L-2(lX+mY)=0,$$

which is the radical axis of the nine-points circle and the circle on BC as diameter. Also its other points of intersection with $S=0$ are the points in which $L-\tfrac{1}{2}L_0=0$ meets it. Hence the equation of the nine-points circle is

$$(l^2+m^2)S+M(L-\tfrac{1}{2}L_0)=0. \qquad \ldots\ldots\ldots\ldots(10)$$

(iv) The equation of the circumcircle (9) may be written

$$(l^2+m^2)S+L\{M-(a+b)L_0\}=0.$$

Hence the radical axis of this and the nine-points circle is

$$2(a+b)L-M=0, \text{ or } (a+b)L+2(lX+mY)=0.$$

Since the circle through the feet of the altitudes is the inverse of the circumcircle with respect to the self-polar circle, the three are coaxal, so that the equation of the last may be written

$$(l^2+m^2)S+L\{M-(a+b)L_0\}+\mu\{(a+b)L+2(lX+mY)\}=0.$$

Also A and the orthocentre are conjugate points with respect to the circle on BC as diameter. Hence the radical axis of this circle and the self-polar circle, *i.e.* the polar of the orthocentre with respect to the former, passes through A; that is,

$$(a+b)L(\mu-L_0)+2\mu(lX+mY)=0$$

passes through (x_0, y_0).

Hence $\mu=L_0$, and the equation of the self-polar circle is

$$(l^2+m^2)S-2(L-L_0)(lX+mY)+(a+b)L^2=0.$$

The radical axis of this and the circle on BC as diameter is $lX+mY=0$, which is the locus of mid-points of chords of $S=0$

perpendicular to BC, by (6). This is clear from Fig. 5, in which the sides of ACK are perpendicular to those of HBC, while the

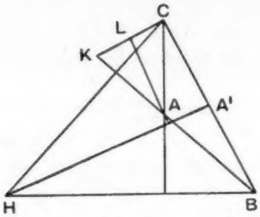

Fig. 5.

radical axis in question, AL, is perpendicular to HA'. Thus L is the mid-point of CK.

4. *The medians.*

The median through A is the locus of the mid-points of chords parallel to $L=0$. Its equation is therefore, by (6), $X/l=Y/m$. This may be written

$$\frac{x-x_0}{bl-hm}=\frac{y-y_0}{am-hl}\equiv\frac{-L_0}{bl^2-2hlm+am^2},$$

for the mid-point of BC.

Hence the coordinates of the centroid are given by

$$\frac{x-x_0}{bl-hm}=\frac{y-y_0}{am-hl}=-\frac{2}{3}\,\frac{L_0}{bl^2-2hlm+am^2}.$$

From § 2 we know the coordinates of the orthocentre. Calling the abscissae of the centroid and orthocentre x_0+x_1, x_0+x_2 respectively, the abscissa of the circumcentre is $x_0+\frac{1}{2}(3x_1-x_2)$, and that of the nine-points centre is $x_0+\frac{1}{4}(3x_1+x_2)$; and similarly for the ordinates.

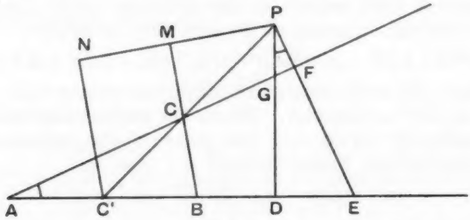

Fig. 6.

We can obtain the combined equation of the other two medians from Fig. 6, in which C' is the mid-point of AB and $C'N$ is parallel to BC. Also P is any point (x, y) on $C'C$ and PMN is perpendicular to BC.

We have
$$PM = PC \sin C'CB$$
$$= PF \sin C'CB/\sin C'CA$$
$$= PF . AC/AB,$$
and
$$PN = PC' \sin C'CB$$
$$= PD \sin C'CB/\sin CC'B$$
$$= PD . C'B/BC.$$
Hence
$$PM . PN/PF . PD = \tfrac{1}{2}AC . AB/BC^2,$$
while
$$PF = PG \cos A.$$
Hence
$$PM . PN/PF . PD = \tfrac{1}{2} \sin B \sin C \cos A/\sin^2 A$$
$$= \tfrac{1}{2} \frac{\sin B \sin C}{\cos A} . \cot^2 A,$$
that is,
$$2 \tan^2 A . \frac{L(L - \frac{1}{2}L_0)}{l^2 + m^2} = \frac{bl^2 - 2hlm + am^2}{(a+b)(l^2+m^2)} \cdot \frac{S}{a+b},$$
or
$$4(h^2 - ab)L(2L - L_0) = (bl^2 - 2hlm + am^2)S.$$

5. *The bisectors of the angles.*

The bisectors of the angle A are $(X^2 - Y^2)/(a-b) = XY/h$. If P is on one of the bisectors of C, PM perpendicular to BC and PFE perpendicular to AC, we have $PF = PM$

and
$$PE - PF = FE = AF \tan A.$$

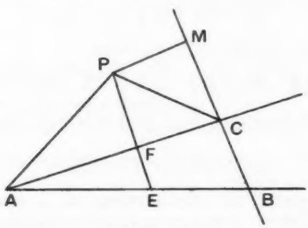

FIG. 7.

Hence
$$(PE - PF)^2 = (PA^2 - PF^2) \tan^2 A;$$
and so
$$(PF . PE - PM^2)^2 = \tan^2 A . PM^2(PA^2 - PM^2).$$
Thus
$$\left(\frac{S}{a+b} - \frac{L^2}{l^2+m^2}\right)^2 = \frac{4(h^2-ab)}{(a+b)^2} \cdot \frac{L^2}{l^2+m^2} \cdot \left\{(x-x_0)^2 + (y-y_0)^2 - \frac{L^2}{l^2+m^2}\right\},$$
that is,
$$\{(l^2+m^2)S - (a+b)L^2\}^2$$
$$= 4(h^2-ab)L^2[(l^2+m^2)\{(x-x_0)^2 + (y-y_0)^2\} - L^2]. \quad \text{...(11)}$$

Now with the usual notation
$$C(x-x_0)^2 = Cx^2 - 2Gx + A, \text{ since } CA - G^2 \equiv b\Delta = 0,$$
$$= bS - Y^2.$$

Similarly $$C(y-y_0)^2 = aS - X^2.$$

Hence $$\{(l^2+m^2)S-(a+b)L^2\}^2$$
$$+4L^2[(l^2+m^2)\{(a+b)S-X^2-Y^2\}+(h^2-ab)L^2]=0,$$

or $$\{(l^2+m^2)S+(a+b)L^2\}^2=4L^2\{(l^2+m^2)(X^2+Y^2)-(h^2-ab)L^2\}. \quad (12)$$

The centres of the four circles touching the sides are the four points (other than B and C) in which these four lines intersect.

For example, if the lines are

$$2xy=0, \quad 3x+4y-12=0,$$

we have, from (12),

$$(50xy)^2=4(3x+4y-12)^2[25(x^2+y^2)-(3x+4y-12)^2],$$

or $$(3x+4y-12)^4-25(3x+4y-12)^2(x^2+y^2)+625x^2y^2=0,$$

or $$\{(3x+4y-12)^2-25x^2\}\{(3x+4y-12)^2-25y^2\}=0;$$

that is,

$$(8x+4y-12)(-2x+4y-12)(3x+9y-12)(3x-y-12)=0.$$

These four lines meet in the six points: $(4, 0)$; $(0, 3)$; $(1, 1)$; $(6, 6)$; $(3, -3)$; $(-2, 2)$, the first two of which are vertices.

It is interesting, but somewhat arduous, to find the tangential equation of the in- and ex-centres. The steps of the undertaking are as follows:

(i) If AX, BY, CZ are perpendiculars to the line

$$L' \equiv l'x+m'y+n'=0$$

passing through one of the centres, then

$$AX \,.\, BC \pm BY \,.\, CA \pm CZ \,.\, AB = 0.$$

Calling these rectangles α, β, γ respectively, the combined equation is

$$\{\alpha^2-(\beta^2+\gamma^2)\}^2=4\beta^2\gamma^2.$$

(ii) If P is the point of intersection of L and L', θ the angle between them, and PM, PN are perpendicular to AB, AC, then

$$BY \,.\, CA = 2R\sin\theta \,.\, PM,$$
$$CZ \,.\, AB = 2R\sin\theta \,.\, PN.$$

(iii) Since $\Delta=0$,

$$(ab-h^2)PA^2=(a+b)S-(X^2+Y^2),$$

and $$(ab-h^2)S=bX^2-2hXY+aY^2.$$

(iv) $$L_0' = \begin{vmatrix} l' & m' & n' \\ a & h & g \\ h & b & f \end{vmatrix} \div (ab-h^2),$$

$$X = \begin{vmatrix} l' & m' & n' \\ l & m & n \\ a & h & g \end{vmatrix} \div (lm'-l'm),$$

$$Y = \begin{vmatrix} l' & m' & n' \\ l & m & n \\ h & b & f \end{vmatrix} \div (lm' - l'm).$$

Calling these determinants δ_1, δ_2, δ_3 respectively, the tangential equation reduces to

$$(\lambda - a)(\lambda - b) = h^2,$$

where

$$\lambda = \{(l^2 + m^2)\delta_1{}^2 + (ab - h^2)(\delta_2{}^2 + \delta_3{}^2)\} / (b\delta_2{}^2 - 2h\delta_2\delta_3 + a\delta_3{}^2).$$

Taking the same numerical case as before :

$$\delta_1 = \begin{vmatrix} l' & m' & n' \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -n',$$

$$\delta_2 = \begin{vmatrix} l' & m' & n' \\ 3 & 4 & -12 \\ 0 & 1 & 0 \end{vmatrix} = 3(n' + 3l'),$$

$$\delta_3 = \begin{vmatrix} l' & m' & n' \\ 3 & 4 & -12 \\ 1 & 0 & 0 \end{vmatrix} = -4(n' + 3m'),$$

and $\lambda = \{25n'^2 - 9(n' + 4l')^2$
$$- 16(n' + 3m')^2\} / 24(n' + 3m')(n' + 4l').$$

Hence $$[-72n'l' - 96m'n' - 144(l'^2 + m'^2)]^2 = 24^2(n' + 3m')^2(n' + 4l')^2,$$

which reduces to

$$(l' + m' + n')(6l' + 6m' + n')(3l' - 3m' + n')(-2l' + 2m' + n') = 0,$$

giving the same points as before. N. M. G.

1181. Carnot published several able works on scientific subjects ; but his literary reputation rests chiefly on his celebrated theory for the defence of strong places, in which, in opposition to Vauban, he strenuously maintains that the means of defence in fortified towns may be made equal or superior to those of attack, so that they could never be taken. His plan . . . rests on . . . a large number of howitzers and thirteen-inch mortars . . . in readiness to open a concentric fire upon any enemy who should attempt to run the sap up to the top of the counterscarp, thus making a vertical fire the basis rather than an accessory to the defence. And he demonstrated, by the calculation of chances, that such a number of these would take effect as to prove fatal to any attacking force, and the larger the more certainty . . . It is not a little remarkable that Carnot's scientific calculations, perfectly accurate if there were no atmosphere, proved erroneous from not taking into account the *resistance of the air* ; just as his political speculations proved so destructive from not taking into account the resistance or impulse of human wickedness.—Alison, *History of Europe*, II, p. 313, footnote.

1182. The Diplomatic Axis.

Geneva might become the fulcrum of the Rome-Berlin axis.—*Manchester Guardian*, November 24, 1937. [Per Mr. C. E. Kemp.]

THE RELATIVE VALUE OF PURE AND APPLIED MATHEMATICS.*

Dr. W. G. Bickley (Imperial College): If I were a Pure Mathematician—which I am not—convention would compel me to preface my remarks by presenting watertight definitions of two nouns—*value* and *mathematics*—and two adjectives—*pure* and *applied*. Actually I prefer to adopt the more natural process, the process by which we have all assimilated the meaning of such terms as " irrational " and " limit ", and let the meanings gradually emerge in the course of my remarks. It may help, however, if I say that I am going to regard applied mathematics as something much more comprehensive than mechanics; that " abstract " is somewhat more appropriate to my meaning than " pure "; and that what I shall challenge is a mental attitude which seems to me quite clearly to imply a mis-valuation.

To begin with I will recount some symptoms which have gradually but insistently forced themselves upon my notice.

(1) A surprisingly large number of students come to us to take an engineering course with rudimentary and sometimes fantastic ideas upon the fundamentals of mechanics.

(2) The proportion of failures in Applied Mathematics at such examinations as Intermediate Science and B.Sc. General is persistently greater than that in Pure.

(3) At the examinations for B.Sc. (Engineering) I find, as do my fellow-examiners, that the questions on what would be termed pure mathematics are better done than those on applied.

(4) When I am lucky enough to get on a short list for a mathematical post, I find myself at the interview outnumbered by self-confessed pure mathematicians by 6 to 1.

(5) When we at the Imperial College need new members of the staff, we find (as do technical institutions all over the country) that it is very difficult to secure men whose mathematical outlook is of the type which is necessary if they are properly to serve the needs of our students.

(6) Allowing for considerable deviations from the mean, there is, in my experience, a systematic and marked difference between the success with students of pure and applied mathematicians.

(7) A comparison of the mathematical standard of the technical press in this country with that of other countries is distinctly unflattering to us.

I do not think that these symptoms can be denied, and I submit that they point to a serious deficiency in our mathematical education, and to an indefensible lop-sidedness in the output of mathematicians. My diagnosis of the causes of this unsatisfactory position may, on the other hand, be open to criticism, and I should hope that succeeding speakers would devote themselves more especially to these causes, and to means of removing them.

* A discussion at the Annual Meeting of the Mathematical Association, 5th January, 1938.

Now it seems quite evident that the trouble begins in the pre-certificate stage in the schools, and I believe that two main causes are at work there. One lies in the fact that with the option of subjects being—and rightly being—as great as it is, one cannot expect a relatively large proportion of candidates to take mechanics. But I also believe that the fact that much of the teaching and examining of mechanics is not in the hands of mathematicians has the effect of reducing the number of these considerably below that which we might statistically expect. I believe that there is more or less general agreement that school certificate mechanics papers too often contain unsuitable and unduly laborious questions, and fail to preserve that reasonable continuity from year to year which is necessary to give teachers confidence in entering their pupils for this subject.

With regard to engineering students, the staffing difficulties to which I have referred often mean that some or all of the teaching of mechanics must be delegated to the engineering staff. Well, I know and respect my engineering colleagues, but—in strict confidence—I do not always find that the careful teaching of the fundamentals of mechanics is one of their " special aptitudes ".

And do you see that in each of these cases—if I am right—there is a vicious circle? This circle has got to be cut, and it is up to all thinking teachers to face the issue, and to find and apply the remedies. The remedy, it seems to me, is a re-valuation, and a change of attitude.

Let us go back to the schoolroom, and take a meditative look round. How many of the pupils there are ever going to succumb to the appeal of pure mathematics as an end in itself? The number is infinitesimal, and even among those who will proceed to a university it is still minute. It follows that we cannot expect abstract mathematics studied for its own sake to make any sort of appeal to the majority—to the overwhelming majority—of our pupils. Nor, if we read the history of mathematics, do we find that the impetus to mathematical advance has come mainly from such a subjective urge; over and over again it has been the desire to do something practical which has led to new mathematical ideas and methods. Apart perhaps from the interest in puzzles, such a practical attitude towards the subject is almost the only one which is going to attract children. Any teacher who will take the trouble to adopt it consistently will find that it works. I have myself been sometimes astounded by the results which can be obtained from human material which the dull school approach had almost spoiled after the realisation had dawned that mathematics is a body of sensible and *useful* information; an astonishment only exceeded by that of finding that the school course seemed to have given no inkling of such an idea. Consistently to adopt this realistic attitude means work for the teacher, of course. He must go outside the school textbooks for his material, and he must know something—a good deal—besides mathematics. Properly to do his job, he *must*, for, to misquote

Kipling, " What know they of Mathematics who only Mathematics know? "

From this point of view, most school mathematics is " applied " mathematics, and many good teachers already use its appeal. Mr. Siddons, for instance, has upon more than one occasion told us how he has used the desire to make scale drawings as the leading motive in an introductory course in geometry. Mr. Inman recently told the London Branch how he used effectively as illustrations of the practical use of the properties of the parallelogram such instances as the locomotive coupling rod and the common mechanism for " ganging " a set of electric switches. Isn't this applied mathematics? So is the application of numerical trigonometry to heights and distances, and to navigation problems. The whole school course, and most of the university course as well, could be similarly treated, and if teachers would take the trouble to do so, and not rest content until they had found such instances ranging over these courses, so that new ideas and methods could be introduced as means to solve problems of practical appeal—well, go and try it out for a year, and return to tell us next January what a vivifying effect it has had on your classes, and to let us know your findings on the relative value of " abstract " and " applied " mathematics!

If all this is, as it seems to me, so painfully obvious, why is it not more generally done?

Well, there are difficulties, of course.

The first is that to be successful we must remain within—unless we can first widen—our pupils' world of realities. Owing to their relative lack of knowledge and experience, this world is limited. From our present standpoint I surmise that girls may be fractionally worse off than boys; but I am not entirely convinced of this.

The second is more complex, and demands consideration of the nature of pure mathematics, or rather of the pure mathematician. He is, psychologically speaking, a freak. His mental make-up is abnormal; he is satisfied without the usual requirements of " practicality " or " reality "; indeed, he often seems to regard his conceptual world as almost more real than, and even as entirely unconnected with, the objective world of the ordinary mortal. Straying into " applied " fields, he has been suspected of denying the relevance of observation to what he alleges to be physical or cosmological theories. But it must be he who advances his subject, and it is difficult to see how he is entirely to be prevented from taking a hand in its teaching. He frequently behaves as though (as he has indeed been overheard to remark) such and such a piece of mathematics is of interest because—*because*, mark you—it is of no conceivable use to anyone! It is comforting to know that most of our pupils have enough sense to resist such opinions, and were it not that mathematics has so many useful jobs to do in the world, I should be entirely content with this resistance.

The innate peculiarities of the pure mathematician are too often intensified by his training. This training is so pitifully narrow.

Any youngster who has mathematical leanings, passing school certificate (as he well may) before he is fifteen, will study seriously in the next six or seven years no subject but mathematics. He will study it in close association with teachers most of whom are, in the sense already explained, abnormal. His range is restricted, since only his intellect is developed. Mechanics and mathematical physics are divorced from the laboratory, and geometry from the drawing-board. Even the "applied" that he may be made to study is vitiated by the artificial nature of many of the questions which he must answer, which are in fact too often "pure" with the thinnest disguise, so that our poor unfortunate may well be excused for the disfavour with which he is apt to regard applied. Finally, if he is not disposed towards the technical applications of his subject, what is left for him to do but teach? Having nothing else to teach, another vicious circle is completed. I urge most strongly that mathematics alone, even leavened by real (as opposed to "sham") applied, is too narrow a training for any teacher.

Further to this, the pure mathematician, if he be nothing but a pure mathematician, must, for reasons which the psycho-pathologist understands so well, retreat further and further into his subjective world and attempt subconsciously to rationalise his conduct. Helped partly by an unjustified connotation of the word "pure", imagining that pure mathematics is to the mind what pure air and pure food are to the body, he proceeds by methods exactly parallel to those of the witch-doctor of an African tribe to create a fetish for the ignorant to worship—and to propitiate by offerings in kind. He has, indeed, done it rather well, for his intelligence quotient is usually high. So well that some applied mathematicians pay lip service to the statement that "God is a Pure Mathematician", which any psychoanalyst knows merely implies the wish to be considered at least a demi-god. This valuation of pure mathematics is out of all proportion, and cannot be accepted.

There is also the lingering of the effete tradition that to soil one's hands with a useful job of work is somehow degrading—a tradition that coined the word "slavey" and is reaping a just if belated reward. Much of applied mathematics, and perhaps especially the applications to engineering, is still fighting this mis-valuation, which leads to many of our engineering students not getting a fair deal at the hands of their mathematical teachers.

Before I close, let one point be clear. It is *not* Pure Mathematics that I challenge, but the unreal valuation which the pure mathematician so often seems to place upon his subject. This, in my opinion, along with its consequences, is seriously jeopardising the success of much of our mathematical teaching. Science, including mathematics, is a social activity, as is also teaching. Now the primary social value of mathematics is negligible—less than that of bridge or chess, shall we say—but its secondary value, in virtue of the many fields of application, is considerable, and is daily increasing. This is the aspect which should illumine our teaching.

Finally, whatever good and sufficient reasons you may individually have, outside mathematics itself, for regarding the subject as a proper and even necessary part of education, it is easy to argue that applied mathematics is at least as efficacious as pure. It has also the inestimable additional advantage that it so much more easily prevents our pupils from making fools of themselves.

Mr. C. G. Nobbs (City of London School) : Dr. Bickley complains that Pure Mathematics is more popular than Applied. I think it is inevitable that there should be better examination performance in Pure than in Applied Mathematics since I believe that in its early stages, at any rate, Mechanics presents greater difficulties than Pure Mathematics at the same stage. For example, I have always found the Calculus and Coordinate Geometry paper in School Certificate easier to prepare for than the Mechanics paper. This may, of course, be due to a discrepancy between the standards of the two papers but is, at least in part, due, I believe, to extra difficulties inherent in the beginnings of Mechanics.

If we begin with Statics we immediately encounter the difficulty of force. It must be a common experience with all of us to find beginners who cannot tackle a question just because they are unable to draw the forces. It takes a great deal of time and practice with many different examples to overcome this difficulty. It is customary nowadays to dodge it by beginning with Kinematics because the difficulties there are easier to smooth over. Nevertheless, however plausibly we may smooth them over, I believe they often lurk, unrecognised, beneath the surface and hamper progress.

The second difficulty is the close interplay of the principles involved, making it very difficult to give practice in the use of one principle at a time. For example, right at the very beginning the idea of Centre of Gravity has to be introduced with the principle of Moments to give realistic examples involving weighty bodies. Compare an average textbook on Mechanics with one on Algebra or Calculus or Trigonometry and, on the whole, more bookwork will be found between successive sets of examples in the former than in one of the latter.

The third and perhaps the most serious difficulty arises from the experimental nature of the subject. This is, of course, an old and thorny problem closely related to the question of whether mechanics should be taught by a mathematician or a scientist.

Dr. Bickley has said that all school mathematics is really applied mathematics. Geometry, for instance, is based upon a large number of experiments made by a boy in his early years and long forgotten. In the same way I believe the study of Mechanics should be based on a boy's intuitional knowledge of mechanical things. It is not possible to arrive at quantitative laws in this way, but it is generally possible to show, by a number of homely illustrations that such laws are reasonable. Consider, for example, the principle of moments. It is easy to carry conviction that the moment increases with the force and with the distance of the fulcrum from the force. The exact

law can only be found by a set experiment. Occasionally, through lack of time, I have omitted this experiment. I have never found any noticeable diminution in the rate of progress resulting from this omission. On the other hand, I have noticed that boys from a science fifth, who have previously had an experimental course in Mechanics, are just as ready as boys from language fifths to find the moment of a force by multiplying its magnitude by any length they may happen to find in the figure, regardless of whether it is perpendicular or not. An experimental course alone does not seem enough. Yet such boys know the principle. You have only to point to their mistake and they say instantly, "Of course. How stupid." It seems as if their knowledge is merely passive—not deep enough to be acted upon.

A few years ago I had a particularly vivid illustration of this. One August Bank Holiday at 8.30 p.m., with one companion, I left Ramsgate in a boat thirty feet long, bound for Flushing. There was practically no wind, and we averaged five knots under engine all the way across. The longest distance to be navigated from mark to mark was thirty-two miles from the North Goodwin Light Vessel to the West Hinder Light Vessel. The tide was setting more or less abeam up the North Sea away from Dover. Before reaching the North Goodwin I went into the cabin and estimating the strength of the tide from the tidal atlas, laid off the usual parallelogram on the chart. The course, thus worked out, I gave to my companion; he swung the helm over and we left the North Goodwin pointing S.E., *i.e.* in the direction of Dunkerque. Down in the cabin I drew the line on the chart along which we were steering—it led nowhere near Flushing. I thought of the vast expanse of gently heaving black water all round us. No doubt it sounds fantastic here in this solid room where we know our position exactly, to say that I was seized with panic. I reworked the parallelogram construction—it seemed quite correct. I had to spend some time resisting an impulse to rush on deck and steer direct for the West Hinder. I had made careful preparations for this voyage—it was the longest we had undertaken—but distrust of the parallelogram of velocities was a quite unforeseen possibility. However, after a good deal of shinning up the mast and anxious peering ahead, we passed one mile to the southward of the lightship.

As a result of that experience I believe that I now have a dogmatic belief in the parallelogram law amounting almost to religious faith. I believe that if anyone offered to demonstrate its falsity I should indignantly refuse to listen. I suggest that here lies the extra difficulty of mechanics over pure mathematics. Our knowledge of number and the laws of number is acquired so early in life that it now has the unshakeable character of belief. The application of the laws of number is instinctive; the application of the laws of forces and velocities is only made after a conscious effort. Laboratory experiments do not bite deep enough; a wealth of vivid experience of the natural laws from the earliest years onwards is a surer founda-

tion. This is difficult or impossible for us to give and is our chief handicap in teaching mechanics. We cannot launch our boys across the North Sea when we wish to prove the parallelogram of velocities. Much can be done by parents on summer holidays, for instance, and by giving their offspring Meccano sets. I always pick out from my mechanics class at the beginning of the year those that have played with Meccano to form the fast section. But every year there is a minority who have had so little experience of practical things that Mechanics is, to them, at least as abstract as Pure Mathematics.

The resultant effect of these difficulties is to make us postpone the introduction of Mechanics into the mathematical curriculum. But I do not believe we shall achieve a proper balance between Pure and Applied Mathematics in schools until Mechanics is introduced much earlier than has been usual in the past. This concerns the School Certificate examinations to which I now pass on.

The Mechanics syllabus in School Certificate is wide and if preparation for it comes out of the mathematics time, two or three periods a week for one year is insufficient to cover it properly except with the very best boys. There are two alternatives : either we can cover a part of the syllabus thoroughly or the whole syllabus sketchily. I prefer the former method, but sometimes I have been badly let down. One year when I had " done " Newton's Second Law, but not Friction, the only question involving Newton's law was one about a stone propelled with given velocity across a sheet of ice whose coefficient of friction was given. My boys were prevented from showing their knowledge of Newton's law by their ignorance of the laws of friction. I am not complaining about the question ; it was easy and within the syllabus. I am simply pointing out that (owing to the close interplay of the principles involved), if the entire syllabus is not covered it is apt to restrict the candidate's choice very seriously. I believe this difficulty of covering the syllabus properly is largely responsible for the small number of candidates who take mechanics in School Certificate. It practically restricts the entry to the good mathematicians.

A possible solution which I am trying now, is to introduce Mechanics into the Mathematics syllabus earlier. At present we begin, in the upper sets of the fourth form, with Kinematics. This work, for some time, is largely graphical. I believe very strongly in giving practice in finding distance travelled and velocity acquired in cases where the acceleration is variable. It takes time, but I do not see any other way of avoiding that common error of supposing that the formulae $s=ut+\frac{1}{2}at^2$, etc. are of universal application. By the time this work is completed, Trigonometry has been started and Statics can be begun. A preliminary discussion of the various types of force with a lot of practice in drawing forces takes the next few weeks and then the law of moments and the law of the resolved part with a few easy examples on centre of gravity complete the year's work.

More can probably be done in this direction. I am considering the introduction of the graphical kinematics into the third form syllabus

and I shall be interested to hear if anyone has attempted this. Curtailing the syllabus is an obvious solution. It is, however, difficult to suggest what should be left out. All questions involving the gravitational and absolute systems of units might certainly be omitted. These do not involve any fundamental principle. Solutions involving either gravitational or absolute units should be allowed. Personally I use gravitational units throughout the School Certificate course and postpone absolute units and the serious discussion of the distinction between mass and weight to the sixth form.

Finally, a larger number of questions giving a wider choice might help to solve the difficulty.

I hope it will be agreed that there is room for improvement in the present position of Mechanics as a School Certificate subject. I think we owe it to the Minister of Transport to launch into the world as many secondary schoolboys as possible with a sound knowledge of the principles governing the motions of fast-moving vehicles. It is also important that the good mathematicians should take the Mechanics paper successfully and easily. At present they often get fewer marks in Mechanics than in the Pure Mathematics paper and this leads to the feeling that they are no good at Mechanics. In this way, Pure Mathematicians are born.

I have not left myself much time to deal with the position of Applied Mathematics in sixth form work. There are two points which I should like to mention.

First, the university scholarship examinations usually consist of two pure papers, and one applied, and one mixed paper. At Cambridge, for example, the applied paper consists entirely of mechanics. I always feel that the number of principles on which this paper is set is really so small that the examiner's ingenuity is largely directed to setting questions in which the mathematics following the application of the principle is sufficiently difficult. If the term applied mathematics included some or all of the subjects, Hydrostatics, Electricity, Light and Astronomy, two pure and two applied papers could be set. This would lead to a better balance in sixth form work. It may be objected that the syllabus is already difficult enough to cover. I agree ; the pure mathematics syllabus would have to be correspondingly reduced. Perhaps we could then have less work on the conic and less trigonometry. A change in this direction would, I am convinced, lead to greater interest on the part of our pupils and an increased sense of the value of the work done. We should do well to follow the example of the classical tradition. The undoubted value of this system of education is, I believe, due to the fact that the languages studied are used as pegs on which to hang all sorts of interesting literary, historical, economic and sociological enquiries. It often seems to me that our mathematical system of education has scarcely got beyond the study of the language.

The other point I would like to mention concerns Dr. Bickley's remark that many of the students sent to him have " rudimentary

and even fantastic ideas of the fundamental principles of mechanics." I have already said that I believe this is partly due to lack of practical experience. I believe it is also partly due to the arrangement of the Higher Certificate examinations. Most of the students proceeding to the science or engineering colleges take the science with mathematics group. Such boys are handicapped as far as their mathematics is concerned partly by lack of time, but also by the low standard demanded by the examination. The alternative to the science with mathematics group is mathematics with full or subsidiary physics. The habit of taking full mathematics with full physics deserves encouragement—perhaps by the creation of a new group—and provides, I believe, the best course for engineers. As long as the easier mathematics papers for science candidates exist, many will take them and many will pass on to the universities with inadequate knowledge of the principles.

Miss K. I. Sayers (Lowther College, Abergele): I am here really under a misapprehension; also I am something of a "gatecrasher". The misapprehension is due to the fact that up to about a week ago I thought the subject of this discussion was to be the relative importance of pure and applied mathematics and only later did I find it was to be the relative value. I once aspired to be a pure mathematician, and I felt that perhaps I could get to a working definition of "importance", not for convention as Dr. Bickley has indicated, but because I think it clarifies the argument, and I am afraid that I am no philosopher to deal with values. It does, however, make the task of speaking slightly easier because one can be so much more vague!

I am a "gatecrasher" because for the past two years I have been administering a boarding school and it has been my recreation and rest to teach the middle school mathematics for about twelve periods a week, and I have not been directly concerned with Higher Certificate and scholarship syllabuses, nor with the balance of pure and applied mathematics.

My remarks are going to be brief, because I think this is a subject which can be dealt with better in discussion around the table than in discussion here. We nearly all of us have our ideas as to whether pure or applied are given their fair places or not. I would rather not differentiate between boys and girls for the moment. I should very much like to see mathematics and physics taught as one combined subject, which would, of course, include mechanics. I cannot myself see the point of differentiating between pure and applied. This combined subject should always be taught by mathematicians, and I would like to go further and make it impossible for a student to read mathematics at the University without reading physics. It would really transfer the physics group to the mathematics, and not *vice versa*. As far as schools are concerned, this would mean that the mathematics and physics staff would be interchangeable; it would not mean that the physics and chemistry staff would be interchangeable, at least not necessarily, nor even the biology and physics.

What would happen to the group of science students reading biology and chemistry, etc., and not mathematics? I suggest that just as it is possible to run a calculus course for these people so it would be possible to run a physics course for them. I have unblushingly taught calculus without tears to biologists and others, who seem to have found it useful, and I am sure the first M.B. physicists and so on could be taught their physics in the same way.

I am a strong advocate of not starting applied mathematics, calculus, etc., until after the age of sixteen, or whenever the School Certificate is taken, although simpler work will often have been used as illustrations. There are very few of us who give a straightforward arithmetic lesson without appealing to something or other in everyday life. The main difficulty of postponing the beginning of these subjects is the economic one ; on the whole, children must be made financially independent of their parents as soon as possible, and therefore they have to start specialising and get through the University as soon as possible. In the general school course a general science course should be an integral part, but the narrowing of the outlook by dropping geography or history or Latin or Scripture to make room for statics or dynamics is deplorable, to say nothing of the practice of dropping art and music which are regarded as frills.

When we get down to " brass tacks " something has to go. We get a boy or girl who has found elementary mathematics easy ; history or geography or Scripture—two or three of these subjects—will go. And that happens at the age of thirteen or fourteen, probably not later than fourteen, because these children will take their school certificates at sixteen.

I would not be dogmatic because here the home life and influence does operate decisively, and if there are good books and wide interests and good conversation in the home, the most narrowing school course will not produce the abnormal mathematician that Dr. Bickley mentioned. But it will throw a good deal of work on to the form master or the headmistress to decide whether it is safe for such and such a child to drop its geography or history.

Personally, I do not think the mathematician is more abnormal than other specialists, say the poet, the musician, the artist, and the author, who are not primarily concerned with contacts with other people. It is when the mathematician turns teacher that the trouble may become apparent. Teaching is essentially a matter of sympathy and inspiration and, with the best will in the world, the unbending mathematician may not be able to establish contact with a dull form or immature minds.

To pass to the problem of girls' schools and applied mathematics, there is no doubt, I think, that girls have had an extra difficulty, in that pulleys and levers, crankshafts and springs, etc., do not form part of their everyday conversation, nor do they wish it to be thought that they do. For myself I happened to be very fond of physics, partly owing to home influences and partly to the fact that my school

had good apparatus. But, nevertheless, when up at Cambridge my questions in applied mathematics were all done as so much pure, and I was deeply grateful for the suggestion whereby nearly all the problems were reduced to differential equations or rates of change of vectors. Later I was coaching women for Parts I and II of the Tripos, and I found how valuable even my year or two of school physics had been, because some of these women had done no physics at all.

To get nearer to Dr. Bickley's point, I differentiate between the routine work and the application, and in this sense pure mathematics would be the routine work, and applied would include all the rest. Before problems of any kind are tackled I try to ensure that the girls can calculate and use the ordinary rules of algebra and arithmetic without much worry. We sometimes swing rather far in the other direction now, and are apt to take for granted that they can all multiply and divide and deal with fractions, even though they have not got this technical skill. I think we should go back a little and improve the technique. With my forms I spend some time in translating without working out anything or getting to any answer at all—particularly translating algebraic problems into equations without solving the equations. Sometimes we try the reverse process, I give an equation and the girls a story to fit it. This cultivation of imagination, accuracy, and ability to read and think is possibly one of the greatest contributions that ordinary school mathematics makes to general education.

I think also that we must not forget that it is of great benefit to the children to have an unemotional subject in their curriculum. They sit down to work out strings of examples in arithmetic rather as a relief from continual discussion of what people did or said, or whether what they thought was right or wrong. You do find that they appreciate the relief. I am sure too that a large number of girls and boys doing elementary mathematics can appreciate the beauty of a simple and neat geometrical proof. But mathematicians in a girls' school, when they begin to specialise, are probably more cut off than mathematicians in a boys' school because there is not that body of non-specialist mathematicians who will be the engineers and technicians of the future which we find in boys' schools. The consequence is that they have to be physically and mentally very strong indeed if they are not to be regarded as something strange, and there is nothing a girl is more anxious to avoid ; their mathematics, except when they are at college, is, as it were, hidden from the world. It is small wonder that their applied mathematics, which depends on general knowledge of things in the world, should suffer. The outlook, I think, is not really as gloomy as this, and even some of those who complete the circle by returning to teaching (or may we call it keeping the lamp alight) have found time to become fairly well-read and to study and give time to their friends and families as well as to acquire the skill required in running a house and bringing up a family, if called upon to do so.

There is this extra problem, that there are probably as many boys

to be educated as girls, and whereas about 50 per cent. of the boys at universities come from public schools a large number of girls come from private schools or small schools where it is the exception to find special teaching in mathematics. The work in the bigger public boarding schools, where teaching and staffing are better, is often handicapped in mathematics by bad ground-work in private schools. I feel rather strongly about this. I am not criticising it, but certainly the supply of mathematical teachers is affected by it.

Finally, may I very humbly suggest that if the move came from the universities to combine mathematics with physics the gap between the present pure and applied mathematicians would not be necessary and we should have co-operation instead of possible antagonism, and that might clear the air as well as the textbooks. The move is always from the top downwards: the entrance and scholarship requirements and syllabus for Higher Certificates would be affected (but I am very hazy what object the Higher Certificate has except as a basis of award of State scholarships). The School Certificate and pre-School Certificate work would be affected inasmuch as only general science and elementary mathematics could be taken, but I would suggest that enjoyable courses for non-mathematicians past the School Certificate stage, taking two or three periods a week, could be run on a par with the present English and French lessons, which we usually expect our mathematics and science specialists to attend. I am trying very hard to say that mathematicians should at least be keeping up with French and with current affairs, and conversely I am sure that there are a large number of girls, and probably boys, who are reading arts courses, who would like to get that bit of insight into mathematics they could get at that more mature stage, without any examination in view. I do know that the staff on the whole enjoy teaching those courses to older children, because they are not tied down to a syllabus. Finally, is it retrograde to plead for more straightforward work in the middle school? Most of the time spent on problem work would be much more profitably spent on English. For the child each problem is a piece of independent research. Nearly impossible if met in an examination out of its context, but just approachable in term-time if he spots that all in Exercise So-and-so are much the same, and at any rate, one or two are worked out in the text.

Mr. R. M. Gabriel (Leeds University) said he thought that Dr. Bickley would agree that he (Mr. Gabriel) was a pure mathematician and also that he was not a neurotic. In spite of his being a pure mathematician, what he had to say about teaching engineering students was accurate. In teaching applied mathematics to people whose main interests lay in mechancial directions the great aim and object was to get the generality of pure mathematics into action as powerfully and as quickly as possible. If one was teaching people who had had three years' apprenticeship in works, and was trying to get the principles of simple harmonic motion into them, at some time there would come the difficult equation $\ddot{x} = -n^2x$, and a student

would ask " Is that x a distance or an angle? " If one attempted to answer, " Oh, it doesn't matter," the teaching opportunity would be gone. The student would find a difference if one wrote θ for x. He suggested that the main object was to get the student to see that x and θ were interchangeable ; in fact, they were trying to get in a pure mathematical point of view as strongly as possible, they did not want to mention whether the thing was a pendulum or a string. It was his own experience of teaching engineers which made him quite deliberate in attempting to get the individual problem apart from the solving of the equations as early as possible. He was proud to hear his old pupil speak on that view, which he had placed before her when he first began teaching.

He thought Dr. Bickley was guilty of serious omission when he made no mention of the pure enjoyment of the neat solution of the pure mathematician. He was sure that the enjoyment of a brilliant solution was not confined to the one or two in whom it was due to neurosis.

Finally, he had no hesitation in recommending the value of the study of pure mathematics. It had this great advantage in particular—it must not be studied for over-long hours. Short hours were almost always needed. Mathematicians had an enormous advantage over chemists and others in that they needed just four hours a day leaving the remainder of the day for the study of the broadening arts. It was a point to be remembered when the value of their subject was considered, and mathematicians need not apologise for it.

Mr. K. S. Snell (Harrow) said that he had much appreciated the excellent introduction of the subject. He had three points to mention. He thought an earlier introduction of teaching in mechanics was of very considerable help. He must plead ignorance in regard to girls, but, at any rate, this applied to boys. It was of definite assistance to their understanding of the ordinary principles of mathematics, and their enjoyment of the subject was evident. He had tried teaching the subject only in what is called the Certificate year, and he was convinced that it should be taught at least one year earlier and approached on graphical and intuitional lines. The best experience he had had was with two forms parallel at the beginning of the year, one set to work on a revision course of elementary mathematics, the other starting on a course of mechanics ; at the end of the year the elementary work of those doing mechanics was better than that of those who had spent the whole time on the revision course.

He considered that they thought too much of the necessity of boys taking the subject in a certificate examination. Why should they not be content, in the case of boys who were mediocre, to do a small amount of mechanics, without it being necessary for them to take the subject in any examination?

As regards connecting the teaching of mechanics with the science teaching, he had the difficulty which he suspected most people had,

that some boys had done an experimental course and some had not. He had met it partly by invading the precincts of the science school with his form to make a few experiments, thus making the subject more real to some of the boys. He thought they should encourage the use of the science school by mathematicians very much more than was generally done.

Mr. M. P. Meshenberg (Tiffins) said that with one exception the speakers were agreed that more applied mathematics should be done, but no attempt had been made to speak of any applied mathematics other than mechanics. He would like to emphasise one point that had been made by Miss Sayers and that was her mention of the practice of setting a certain number of problems in algebra without asking for the resulting equations to be solved. He found that practice was spreading ; he thought it was Mr. Siddons who years ago made the suggestion that in geometry a considerable time could be saved in geometrical training by getting pupils to read the geometry questions, draw the figures, and then pass on. He had made a good deal of use of that in mechanics—read the question, draw the figure, insert all the forces, obtain the equations and do not stop to solve the equations.

But every problem a child had in algebra, arithmetic or geometry was applied mathematics. Pure mathematics had little to do with the solution of such problems. The problems one met with in elementary textbooks of pure mathematics concerned the application of principles. If the field were extended to include bio-mathematics, on which there was a textbook which gave an admirable survey, and statistics, a wide field could well be tapped and used.

He had come to the conclusion that what Dr. Bickley was getting at was that he would like his mathematics not to be practical so much as topical. Dr. Bickley was obviously obsessed—much more than the neurosis which he imputed to the pure mathematicians—by the mental limitation of his field of vision to wheels within wheels going round and round occasionally with rods connecting them. If they could agree to widen the field from which problems were drawn so as to include topics of the day there was no chance of countering what Dr. Bickley said.

Mr. E. J. Atkinson (Newport, Mon.) said it had struck him that the real cause of the trouble with applied mathematics was that they were brought up to regard it as a difficult subject : in his early days (as was the case in many schools even now) it was thought to be something to be tackled post-certificate. Even if they looked at the London Examination, with which he was familiar till eighteen months ago, it would be seen that mechanics could be a science at School Certificate or it could be a mathematics subject at Higher Stage. He had now to face the Oxford Certificate in which every candidate must take mechanics as part of the physics paper. It was thus taken away from the mathematics side.

Mechanics could be, and he felt should be, introduced from a practical standpoint. In his former school for a year he took the

second year physics and found that even at that stage, starting with dynamics, the boys could enjoy the subject, and they were able to understand what was to come later in applied mathematics. It was not just an abstract subject. He would admit that as a boy he hardly knew what a differential pulley was. Having attempted to demonstrate with apparatus each principle as it came along, he quite agreed with Mr. Snell that if one developed the work from the practical side, the boy who had done some practical work in the early stage of mechanics or applied mathematics could apply these principles to the problems much better in the applied mathematics stage. Therefore the subject was at the present time, in most schools, not the nightmare it was formerly.

Mr. F. J. Swan (Hackney Downs) said he would like to take up one point in particular, that of the technical school. He mentioned the case of one boy who in 1925 in the matriculation stage failed almost completely to understand any mathematics. In 1931 he met the same boy again in a course at a Technical Institute in London where he was taking the Higher National Certificate in engineering, in which the mathematics demanded was very considerably beyond the School Certificate standard. He suggested that this was acquired at the technical school because the boy had an urge.

With regard to one of the remarks of Mr. Gabriel, he would suggest that the questions which were put by the engineers would not have been put except by those who had practical experience As to mechanics, he felt that much of the teaching of mechanics in secondary schools had been simply teaching another branch of pure mathematics. He thought that the trouble in teaching their school mechanics, was that it was divorced from actuality. He challenged anybody to say that some of the problems set in the certificate examinations in applied mathematics were practical. Many of them could not possibly have any practical application.

Another question was that of the marks in the mechanics examination. He believed that failure was often due to the fact that the candidates were asked for descriptions. At the age of sixteen boys were not very apt in their descriptions and he suggested that a proportion of the marks disappeared when they were asked to describe.

Mr. Hope-Jones (Eton) said he thought that geography had been mentioned only once, as one of the subjects to be left out to make room for something else. He believed it had a higher mission than that, and that mathematics and geography should not be divorced from each other. If they could combine mathematics with geography it would be one of the most interesting applications of mathematics and would be an answer to the child who asks, " Why am I learning this stuff? " Another very important application of mathematics was to probability. He believed if their children could leave school with a better idea of the principles of probability less money would pass from their hands into the hands of thieves.

Mrs. Linfoot suggested that one opportunity of broadening the outlook of the teacher of pure mathematics was provided by the year's training which many graduates now received. In her course of lectures in the Education Department of Bristol University she had to deal with a class consisting of both scientists and mathematicians, and had tried to give advice on mathematics courses suitable for science specialists. She would be glad to hear the views of anyone interested in such courses.

Mr. C. G. Paradine (Battersea Polytechnic) said the translating of practical problems into mathematical problems was one of the things lacking in older textbooks. He thought much more ought to be done in that way. The teacher now had to do that translation. He thought the mathematical teacher had got to learn some engineering and physics, and so on, and put these problems in mathematics before the students, otherwise it did not get done. He found in the intermediate stage of mechanics he could get better examples from the examination papers of the Institution of Mechanical Engineers than from the textbooks.

Dr. W. G. Bickley in a brief reply to the discussion said the remarks made had ranged over a wide field. He thanked Mr. Nobbs for his analysis of the difficulties inherent in mechanics. He was also very glad to welcome Miss Sayers's suggestion that mathematics should be more often combined with physics in the Higher School examination, and he believed it should be in the university course also. The appeal made for an increase in technique he sympathised with, but they were talking about " value " and value seemed to imply purpose, and purpose was the root of what he had said earlier. He asked what end the increased technique was to serve. The approach via the applied problem meant that the pure mathematics was going to be better done and better understood, and that way it would increase the power of understanding the physical meaning of the mathematics. He was pleased that Mr. Snell and others had agreed that the early introduction of applied mathematics in the schools did have that effect and did improve performance in pure. With regard to Mr. Meshenberg's remarks, he thought he had made himself clear that he did advocate a wider range of application. With regard to the aesthetic point and the joy in a neat solution, that he was sure was a strong urge, but he regarded it as an individual, rather than a class, matter.

1183. Everyone ought to know how to read and how to write, and to do sufficient simple arithmetic for the ordinary business of life, but beyond that I believe that we have no right to go. If we insist on other subjects we must at least be able to explain what use they can conceivably be. Algebra, for instance, is taught in England to every boy, and to some, no doubt, is of great value, but can anyone maintain it has that value for all? For my own part, I have never had the faintest idea why I was taught it, nor have any of my preceptors been able to suggest a satisfactory answer.—Dr. Cyril Alington in the *Sunday Times*, November 21, 1937.

[Per Rev. J. J. Milne ; Mr. P. J. Harris.]

THE NEW TEACHING COMMITTEE.

MEMBERS will recollect that at the Annual Meeting in 1937 it was agreed to replace the Boys' Schools, the Girls' Schools and the General Teaching Committees by a single Teaching Committee representing teachers of mathematics in all schools and institutions. From this Teaching Committee it would be possible to form small sub-committees competent to undertake inquiries, prepare reports or deal with any similar work suggested by members of the Association. These sub-committees would have considerable power of co-option from outside the Teaching Committee.

The first meeting of the new Committee was held on 5th Jan. 1938, and it seems desirable to enlist the interest and help of members of the Association in the future work of the Committee by publishing some account of the proceedings.

Mr. A. Robson is the Chairman and Mr. C. T. Daltry the Secretary of the Committee.

It was agreed to appoint sub-committees to consider

(i) Examinations

Besides dealing with criticisms of questions there is much constructive work that might be done, for example in considering the possibility of unifying examination syllabuses.

(ii) Revising the Report on the Teaching of Mathematics to Technical Students.

(iii) The presentation in the class-room of the historical side of mathematics.

(iv) A report on the teaching of mathematics to children up to the age of 11.

Three other topics were discussed ; the preparation of reports on the teaching of trigonometry, on teaching the complete duffer, and on the contact of mathematics with other subjects. It was suggested that instead of reports on the last two topics a series of articles on each might be published in the *Gazette*.

Members who are interested in any of the above, and who might wish to contribute to the work of the Teaching Committee are invited to write to the Secretary.

C. T. DALTRY.

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1184. I think of him sitting for two years with a blank sheet of paper before him whilst the *Principia Mathematica* germinated in his brain, day after day sitting at his desk, getting nothing down on paper, yet this mighty work, unfolding, disentangling, emerging into order from chaos, inchoate, yet gradually taking form. It took him and Prof. Whitehead ten years to write this book and the completed manuscript was so vast that it had to be taken to the publishers in a hansom cab.—Ethel Mannin, *Confessions and Impressions*, p. 282.

[Per Mr. J. B. Bretherton.]

THE RELEVANCE OF MATHEMATICAL PHILOSOPHY TO THE TEACHING OF MATHEMATICS.*

BY M. BLACK.

THE comparatively modern subject of mathematical philosophy is the expression of only one of the many purposes with which philosophy is concerned. It is in attempting to provide a " critical examination of the grounds of our convictions, prejudices and beliefs " that philosophers have been led to act as critics of science in general and mathematics in particular. And since understanding must precede criticism, the critical philosopher is even more concerned with describing and analysing the nature of mathematics than with passing judgments upon its achievements. The comparative modesty of this aim, of analysing the nature of mathematics, gives to the researches which it has inspired a definiteness and objectivity which sets the subject far beyond the reach of the everlastingly unsettled disputes of traditional philosophy.

It is unnecessary to enlarge upon the relevance of this undertaking to the teaching of mathematics. In teaching any subject it is as well to know as clearly as possible what it is that one is attempting to teach, and the existence of a large body of researches designed to answer this very question is a fact which no teacher of mathematics can afford to ignore.

Nevertheless it would be highly misleading to suggest that mathematical philosophy can supply a tabloid answer to the question of the nature of mathematics. Indeed it is one of the lessons of the great advances made in our times by the sciences that to a simple question we must never expect any but the most complicated answers. To the question, What is light? for example, modern physics is unable to reply except by directing our attention to the theory of light, a complicated system of interconnected properties. In calling light a kind of electromagnetic vibration we are saying very little unless we mean to refer the enquirer to the whole of modern theoretical physics in which the theory of light finds a place. In other words the answer to an enquiry concerning the nature of light is not a sentence but a complex system of relations, constituting a theory which, because it is ultimately relational in character, is also mathematical in form. If we wish to be as scientific about the nature of mathematics, a phenomenon considerably more complex than any known to physics, we must expect no neat formula as an answer. The kind of insight to be obtained will consist rather of deeper appreciation of such matters as the connections between various parts of mathematics, the mutual interdependence of the basic notations of geometry and arithmetic—results which, as we shall see, can themselves be expressed in mathematical form. I need not dwell on the wider philosophical implications of this programme: philosophers wish to analyse and criticise mathematics with a view

* Annual Meeting of the Mathematical Association, 5th January, 1938.

ultimately to relating it to other types of knowledge. But the unphilosophical mathematician may be sufficiently rewarded by the deepened insight into the structure of his subject which these researches have afforded. The kind of contribution which mathematical philosophy has made to the understanding of the nature of mathematics is best appreciated by tracing the outlines of some investigation in considerable detail. For the purposes of this paper it will be convenient to follow the emergence throughout the nineteenth and early twentieth century of an ideal of mathematical form. We shall be able to see how the emergence of an ideal of mathematical form depends upon an increasing tendency to apply mathematical methods to logic and a correlative tendency for pure mathematics to remove from its province all theories which are not strictly logical in character. We shall see that the tendency for mathematics to become more logical while logic itself becomes more mathematical is accompanied by and depends upon increasing awareness of the importance of symbolism and produces a constant evolution in the very ideal towards which the whole movement is tending.

One of the landmarks in this process of interpenetration of mathematics and logic is the invention by George Boole, in 1847, of an algebra of logic, a system in which the principal ideas of logic are replaced by symbols and processes of reasoning are converted into specified and unchanging rules for the manipulation of symbols. The attempt to apply mathematical methods to logic was not altogether a novelty. The notion of a calculus, a system of unalterable rules mechanically applied, is already implicit in the logic of Aristotle, but the absence of suitable symbolism leaves Aristotelian logic a basis for, rather than an example of, mathematical method. Leibniz, however, two hundred years before Boole, had planned in considerable detail a calculus of reasoning, with appropriate symbols and principles of manipulation, a kind of logical machine to substitute calculation for argument. Leibniz' unsuccessful attempt to carry this project into execution was followed by others (equally unsuccessful however). Many a mathematician, impressed by the contrast between the unanimity of mathematicians and the lack of it in those who are not, has experimented with a similar project; the notion of replacing the conflict of opinion by the unemotional and infallible mechanism of a logical machine recurs constantly in the history of ideas. Boole was sufficiently ingenious or sufficiently lucky to overcome the technical difficulties which had defeated his predecessors and to invent a system which has been the basis of all subsequent research in the field of symbolic logic. It will be worth our while to describe Boole's system in some detail, not only for its intrinsic interest, as one of the outstanding examples of the successful application of mathematical method, but also because the simplicity of the theory which he invented reveals, unobscured by irrelevant detail, those features of an ideal mathematical system which will serve as a convenient text for our subsequent discussions.

In Boole's system, then, we use variables, x, y, z ..., not, as in middle school algebra, to represent integers, but as representations of sets of objects, not necessarily numerical in character, which satisfy some description. Thus the phrase " all the mathematicians in this room " is a description of a certain set of objects ; in Boole's own phrase, we can " elect " all the mathematicians in this room for our consideration and refer to them collectively by x. Logicians are accustomed in such cases to speak of the *class* of mathematicians in this room (rather than to use a word like set or group or collection). Thus all the mathematicians in this room constitute a class denoted by x and Boole's system is an algebra of classes.

The operation of election or selection by means of which we isolate a class and denote it by a single symbol can be repeated and furnish us with a sub-class. Thus let " y " = " all things which are more than five feet high ". If from x, the class of all mathematicians in this room, we select those members who are also members of y, the class of objects more than five feet in height, the result, viz. the class of objects which are *both* mathematicians in this room *and* more than five feet in height, is symbolised by $x \times y$. It will be at once apparent that no matter what x or y may denote, in this system $x \times y = y \times x$.

So far we are in agreement with the laws of school algebra. But if we multiply x by itself in Boole's algebra we shall remain with x : the class of wet days which are also wet days, for instance, is just the class of wet days. Thus $x \times x$, or $x^2, = x$, and, in general, $x^n = x$. We shall soon find other respects in which Boolean algebra differs from the algebra of integers.

For we can easily define logical addition by taking $x + y$ to mean the class obtained by combining the members of x and y into a single class : in the interpretation of our chosen example, $x + y$ would mean the class of objects which are *either* mathematicians in this room *or* are more than five feet high (or both). Now while the definition of logical addition shows that for all x and y we must have $x + y = y + x$, that is to say, logical addition is commutative, like the corresponding operation in school algebra, a little consideration shows that for all x, y, z the following laws are valid :

$$x \times (y + z) = (x \times y) + (x \times z) \quad \text{and} \quad x + (y \times z) = (x + y) \times (x + z),$$

so that both logical addition and logical multiplication are distributive with respect to each other and we have, in contrast to ordinary algebra, complete duality between the two operations.

In order to complete the apparatus of Boolean algebra we shall need, in addition to the variables already described, two constant symbols : 1, which denotes the class consisting of everything, and 0, the class consisting of nothing. In accordance with these meanings we must have, for all x,

$$1 \times x = x, \ 1 + x = 1, \ 0 \times x = 0 \quad \text{and} \quad 0 + x = x.$$

Finally we need a way of representing the class of things which

remain behind when we select x: we use the symbol $-x$, so that, for all x,

$$-x+x=1 \quad \text{and} \quad (-x)\times(x)=0.$$

This completes the description of the notation of Boolean algebra (whose details I have altered in one or two respects in accordance with improvements made since Boole). We could now proceed to take those properties of classes which I have already illustrated in the commutative and distributive laws, and other further properties, use the statements in which they are expressed as axioms, and proceed to make deductions and discover theorems. There is not time for me to develop this calculus in more detail or to show how it can be used for the solution of simple logical problems. But enough has been said perhaps to indicate in outline the general character of Boolean algebra.

The great interest of Boolean algebra arises from the fact that it is one of the earliest examples of pure mathematics which is not tied to a numerical interpretation. For it *is* a branch of pure mathematics though its laws differ in important respects from the algebra taught in schools. The fact that we have understood the variables x, y, z to refer not to numbers but to classes of objects in no wise destroys the mathematical character of the system. For once we have established the axioms of Boolean algebra we can safely forget the interpretations of the symbols, just as in school algebra we need not refer to the numerical interpretation of variables until we are called upon to find some numerical value involved in a practical application. Here we have an important characteristic of any mathematical calculus upon which its use as an instrument for economising thought depends. We construct mathematical systems in such a way as will allow us to forget the meaning of symbols in order to manipulate them by determinate rules, like pieces in an esoteric game.

This capacity for temporary abdication of meaning which Boolean algebra exhibits in common with ordinary algebra has the important consequence that we are not restricted to any one particular interpretation of the symbols. In Boolean algebra, for instance, we can easily give a geometrical interpretation of the symbols involved. By 1 we mean some definite two-dimensional region, *e.g.* a rectangle; by x, y, z, any regions wholly contained in 1; by $x\times y$, the region common to x and y; by $x+y$, the region obtained by coalescing x and y; by $-x$, the region which, combined with x, would make up 1, by 0 the null region (Fig. 1).

It can be verified that all the axioms of Boolean algebra, and, in particular, the laws of distribution and commutation previously mentioned, are also satisfied by our geometrical interpretation. Thus Boolean algebra can be regarded either as expressing the properties of classes or, alternatively, as stating the topology of certain two-dimensional regions. Nor are we restricted to these two interpretations: among others we can also find valid numerical interpretations and thereby remove the only possible reason for drawing

a sharp distinction between Boolean algebra and the algebra of integers. We must speak of algebras not algebra, and recognise that algebraic systems can be erected in independence of numerical interpretations. This enlarges the scope of mathematics in two ways: we are free to invent, as practical problems demand, a new calculus for a new situation, and then, in virtue of the abdication of meaning inherent in such a calculus, express at one stroke the laws and properties of as many different subject-matters as we can find interpretations, in this way unifying departments of empirical science or mathematics itself which might otherwise remain disconnected. We can refer to a system in which abdication of meaning has occurred as *abstract*: it is the abstractness of mathematics which determines its generality, as we have just seen, and can be made the basis of effective analogy.

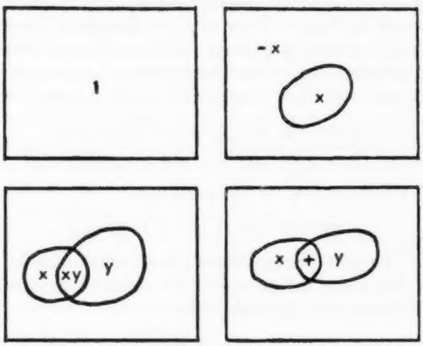


Fig. 1.

To the kind of shape taken by Boole's algebra (and imperfectly attained long before by Euclid's elements) various names have been given according to which aspect of the pattern has most impressed the commentator. Axiomatic, abstract, deductive, hypothetical, formal—all these words refer to distinct but connected aspects of such a system. Axiomatic because based upon a number of statements arbitrarily chosen as axioms; abstract because reference to the meaning of symbols is banished; deductive because no appeal is made to any principles except those contained in the axioms; hypothetical because we explore the consequences of the axioms without needing to assume their truth; formal because we pay attention only to the pattern of combinations. These characters of a theory in pure mathematics have come to be taken so much for granted that I may seem to be stressing the obvious in bringing them once again to your notice. We shall see in a later part of this paper how recent researches in mathematical philosophy have revealed limitations restricting the possibility of achieving a completely abstract system to embrace the whole of mathematics, and

the significance of these latest discoveries is not to be fully appreciated unless we emphasise the advantages of generality and economy of thought of which they threaten to deprive us.

Let us return, however, to the advances in formal logic of which Boole was the forerunner. While Boole's algebra of logic might serve as a model of the presentation of logical principles its content did not extend far beyond the traditional syllabus of logic which had remained unchanged in essentials since its classical formulation by Aristotle some two thousand years previously. In the short period of time since Boole, logic has seen more discoveries than in the whole of its previous history. Here, as indeed throughout the evolution of mathematical philosophy, the driving force has been largely a growing realisation by mathematicians of the inadequacy of traditional syllogistic logic to express the form of some of the simplest and most common mathematical arguments. Consider a simple argument such as "If x, y, z are positive integers such that x is greater than y and y greater than z, then x must be greater than z." Aristotelian logic transforms every argument into the form of a syllogism such as that in which from two premisses :

some (or all) A is (or is not) B,
some (or all) B is (or is not) C ;

there follows a conclusion of the form :

some (or all) A is (or is not) C.

If we attempt to put our mathematical argument about relations of inequality between integers in this form, however, we shall be forced to introduce some premise like

every (number which is greater than y)
A
is
(greater than every number which is less than y)
B

in which we have wrapped up in the places shown by A and B just those properties of the relation "greater than" upon which the argument depends. Thus while it is possible to express every mathematical argument in the syllogistic form it is possible to do so only at the cost of leaving unsymbolised precisely those formal relations upon which mathematical deduction depends. Syllogistic logic is, in short, insufficiently complex for the task of representing mathematical deduction. Now the very fact that Boole's system was so largely mathematical in character and thus independent of any metaphysical views concerning the ultimate nature of logic cleared the way for Peirce, Schröder and Russell to generalise traditional logic. It is noteworthy that the technique of the generalisation involved a separation of the idea of mathematical function from purely numerical interpretation. Introduction of the notion of pro-

positional functions whose values, when the system in which they occur is interpreted, are allowed to be propositions, allows the invention of the so-called calculus of propositional functions which provided a new, more complex, logic, adequate to represent the processes of argument actually used in mathematics and the empirical sciences. It is difficult to exaggerate the importance of this application of mathematical method to the elaboration and formalisation of logic. Logic became, in the course of a revolution comparable only to the invention of coordinate geometry in mathematics, not only an exact, but a useful science.

While this exciting enlargement of the scope of logic was occurring, pure mathematics itself was passing through a period of prolonged self-criticism. The wonderful sequence of mathematical researches in the eighteenth century, made possible by the invention of the calculus and stimulated by great discoveries in applied mathematics, often employed methods which were logically indefensible. The great Gauss, himself a leader in the movement towards greater rigour, employed in his astronomical work terms from series whose convergence had not been investigated. Profound mathematical advances were accompanied by lack of clarity regarding such fundamental notions as complex numbers, infinite series and limits. The resulting confusions were of more than theoretical importance, and resulted often in the promulgation of false results or the use of invalid demonstrations. Gauss and Cauchy were among the greatest of those who, early in the century, preached a return to more rigorous methods, to the ancient logical ideal of the Greeks—a programme which was so far from being trivial, however, that it required a century of mathematical invention to demonstrate how far we still are from achieving it.

We have seen that the reduction of mathematics to strict deductive form is independent of a numerical interpretation of the variables involved in the system ; nevertheless it is characteristic of the efforts made throughout the nineteenth century to improve the rigour of mathematics that they involve the attempt to express as many mathematical notions and processes as possible in terms of integers. For although the expression of a mathematical theory in numerical terms is only a half-way house towards the full emergence of deductive form, the properties of numbers are so well known that a numerical interpretation of a mathematical theory often reveals the form of the theory more instructively than a premature recourse to axiomatisation. Since Descartes' arithmetisation of geometry, the translation of a theory into geometrical terms provided a useful method for the ultimate expression of the theory in purely numerical terms.

Such translation of mathematical theories, whether directly into numerical interpretation or, intermediately, into geometry, has proved a most fruitful method for eliminating the confusions surrounding the fundamental mathematical notions. So long as we think of complex numbers, to take one example, as numbers which

in some mysterious way both do and do not obey the laws of common integers, their use is bound to involve confusion and fall short of conviction. But with Gauss' translation of the properties of complex numbers into geometrical terms, the essential step is taken towards the modern view which regards complex numbers simply as couples of numbers subject to certain specified laws of combination. With this final step, which is, one ought to add, a comparatively recent development, the mystery surrounding complex numbers finally disappears, and they can be employed with full confidence in the consistency and accuracy of the demonstrations in which they occur.

The translation of mathematical notions into properties of entities definable in terms of integers is particularly noteworthy in the development of the theory of functions during the nineteenth century. The differential calculus arose from the consideration of velocities and areas, that is to say, from the sensible properties of sensible bodies, and arguments based upon such properties had been freely used in the advancement of the subject. Consider, however, such a simple statement as " If a function of x is continuous between $x=a$ and $x=b$, and takes negative and positive values respectively at these two values it must take the value zero for some value of x between a and b."

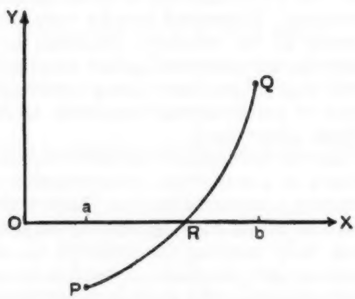


FIG. 2.

So long as we think of the function as represented by a curve which we can " see " to be continuous, we can also " see " that the curve, being below the x-axis at P and above it at Q, must cross the axis at some point R between P and Q. But what can it mean to say that we " see " a function to be continuous? The most we can literally see is a chalk track or an ink trace which has that property—the property of not presenting visible gaps—which in the case of a sensible body we call continuity. But the mathematical curve cannot be seen, except in a metaphorical sense, either by the microscope or by the light of faith. Cruel and renewed experiences with textbooks has taught most of us to interpret the words " it

can be seen that . . ." as a signal of the author's incapacity to present, or our own inability to understand, a strict mathematical argument.

So long as we depend upon people to see, or see through, the properties of imperfectly drawn chalk tracks, we may expect disagreements and mistakes arising ultimately from the lack of definition of such terms as function and continuity. The attempt to *define* continuity, however, naturally leads, as it led Cauchy, to consider a function, not in the form of a geometrical pattern, but in the more abstract and arithmetical form of two correlated sets of numbers. When we define a function in such arithmetical terms, definitions of continuity, of differentiability, of uniform convergence, first become possible. In this way the theory of functions becomes simply a very complicated branch of arithmetic. In such arithmetisation, to use a clumsy but convenient word, of the theory of functions there is both gain and loss: loss in immediacy of presentation, in compendiousness; gain in accuracy and consistency. The renunciation by mathematicians of reliance upon geometrical models is parallel to the effort which physicists have found it necessary to make to abandon the use of mechanical models. The properties of little rigid spheres have been found to be unsound guides to the behaviour of the smallest portions of matter, and just as renunciation of such analogies has permitted physicists to discover properties of matter for which they have no simple images, a judicious sympathy for abstractions has allowed mathematicians to assimilate discoveries (such as that of Weierstrass, of an everywhere continuous but nowhere differentiable function) for which there is no material representation.

The arithmetisation of mathematics in the nineteenth century (which reached an almost pathological fervour in Weierstrass and Kronecker in the seventies) is closely connected with the ideal of mathematics as an abstract system to which I have already called attention in connection with Boole's work. Weakening the bonds which united the theory of functions and geometry at the same time promoted the wider applicability of analysis; theories could be developed which had no sensible equivalents in our own two-dimensional space and yet found application in the sciences. But two things remained to be done before the ideal of mathematics as a completely abstract system could be attained. It was necessary, if possible, to define the properties of the positive integers, upon which the whole of pure mathematics now rested, either by explicit definitions or else by exhibiting a set of axioms from which all their properties could be deduced. Secondly, it was necessary to formulate the rules of mathematical argument. Only with the satisfaction of these two requirements could Leibniz' dream of mathematics, as one huge calculating machine in which neither mistakes nor disputes were possible, be achieved.

By the end of the nineteenth century the reduction of pure mathematics to arithmetic and the consequent unification of the whole of

mathematics into one gigantic system was so far advanced that Peano could undertake the task of formalising arithmetic itself. With the help of collaborators he was able, in his famous *Formulaire de Mathématiques* (1895-1908), to exhibit the axioms upon which arithmetic itself is based, and by using the generalised symbolic logic which I have previously described to exhibit the whole of pure mathematics in strict symbolic form.

The results of this period of evolution had been to bring the subjects of mathematics and logic very close together : the new logic of the nineteenth century had taken on a mathematical appearance, while mathematics itself had gone far in the direction of exhibiting itself as a completely abstract, completely deductive, system. But something still remained to be done in improving Peano's enormous task of synthesis. Russell and Whitehead in their *Principia Mathematica* published in the early years of our own century after nearly twenty years' work, set a new standard of logical precision. Their work extended far beyond the limits of Peano's programme ; in addition to making important improvements and innovations in the theory of propositional functions and the detailed arithmetisation of pure mathematics, work in the upper stories of the mathematical edifice, Russell and Whitehead examined the foundations and showed that the integers upon whose properties the whole of pure mathematics had now been made to depend could themselves be defined. It must be hardly credible that notions as fundamental as one, two or three can themselves, without circularity, be defined. The curious must be referred to Russell and Whitehead's great book, which will remain for all time a noble monument to man's desire for system and order, for the demonstration. The point which is of prime philosophical importance in the system of *Principia Mathematica* is that the primitive notions in terms of which the axioms are expressed belong to logic. From a few notions like *proposition*, *not*, *implies*, *class*, *all*, and a very few axioms, it is possible, or almost possible, to erect first arithmetic in its entirety and then, in turn, the theory of functions and the whole of higher mathematics. Whether this programme was completely achieved is a point as to which opinions differ and upon which we must reserve judgment. At any rate *Principia Mathematica* claims to have succeeded in breaking down the last distinction between logic and mathematics, and succeeds to a positively miraculous extent in substantiating its claim. If we could accept the claim we should be able to say that mathematics is simply logic elaborated and complicated by the introduction of chains of definitions, while logic is simply one of the least complex branches of pure mathematics. Any separation between the two subjects would then be dictated by practical convenience—the fact that we happen to be more interested at any given moment in those complicated parts of logic which we call pure mathematics.

The significance of *Principia Mathematica* in the history of mathematical ideas cannot be adequately assessed in terms of the success or failure of its programme of reducing mathematics to logic. That

programme is only the climax of a movement towards vigorous mathematical method which we have traced in the *rapprochement* of mathematics and logic during the nineteenth century. Deficiencies in *Principia Mathematica* (which, paradoxically enough, only its well-nigh complete success was able to reveal) are defects inherent in the contemporary version of the ideal of a mathematical system. The difficulties at whose existence I have hinted are associated with certain notorious contradictions, of which the most famous (on account of its mathematical character) was invented by Russell himself. Within the limits of this paper it will not be possible to present a sample of these paradoxes expressed wholly in mathematical terms, but I would ask you to bear in mind that paradoxes similar to those which I am about to describe can be put into mathematical dress and that the apparent triviality of the examples conceals an issue of sufficiently vital importance to mathematics to have attracted to its solution the energies of some of the greatest living mathematicians.

To begin with a non-mathematical example, then. Consider the adjectives in the English language. We can divide them into two mutually exclusive classes by considering whether the property expressed by an adjective does, or does not, apply to the word itself. "Short", for instance, is a short word; "English" is an English word: both these words belong to the first class, the class of adjectives which apply to themselves. On the other hand, "long" is not a long word; "German" is not a German word: both these words belong to the second class, the class of adjectives which do not apply to themselves. Let us coin two words to denote this distinction: all adjectives which *do* apply to themselves are to be called autological, and all which do not, heterological. Clearly, every adjective must either apply or not apply to itself; must belong to one or other of the two defined classes. So far we seem to have done nothing contrary to the rules of logic and our procedure appears quite harmless.

Let us now consider the word "heterological" itself. Since it *is* an adjective, by definition, it must itself be either heterological or autological. Let us examine the alternatives separately.

(1) Suppose the word "heterological" is heterological. By definition of the meaning of heterological this would mean that the adjective heterological does not apply to itself, *i.e.* "heterological" is autological, contrary to our assumption.

(2) Suppose then that the word "heterological" is autological. By definition of the meaning of autological this would mean that the adjective heterological does apply to itself, *i.e.* "heterological" is heterological, contrary to our second assumption. Thus whichever assumption we make about the word "heterological" forces us to make also the contrary assumption: we are faced with a contradiction.

Consider again, to take another example, the set of positive integers which can be defined in the English language by less than thirty

syllables. The number of syllables at our disposal is finite, for we are not allowed to invent new words. Suppose the number of syllables is s. There will be a finite number of strings of syllables each less than thirty syllables in length. Some of these strings of syllables will be nonsensical, others will make sense but will not define positive integers, but there will be a certain number of them which do define distinct integers, and there will be a certain definite number, k say, of positive integers each defined by at least one of these strings. Let $a_1, a_2, \ldots, a_k$ be the positive integers thus defined. Consider the smallest integer, N say, which is not included in the set $a_1, a_2 \ldots, a_k$. In words, N is "the least positive integer not definable in the English language by less than thirty syllables". This phrase defines a unique integer, namely N. But the defining phrase itself has less than thirty syllables (even on the most generous calculation). Thus N *is* defined by less than thirty syllables, and this is contrary to its definition as the least integer not so definable. Once again we have a contradiction.

It is easy enough to see that something must be wrong with the arguments used in the formulation of the paradoxes; the difficulty is to isolate the features which render the arguments invalid.

We can observe a certain similarity in the paradoxes which I have used by way of illustration. In the first, the crucial words, "heterological" and "autological", were defined by testing all adjectives without exception. We supposed that every adjective could be submitted to the criterion of either expressing or not expressing a property of the word itself and it was precisely because the word "heterological" was itself an adjective that we were forced to apply to it the same criterion and thus obtained a contradiction. In other words, we defined heterological by reference to a class, the class of adjectives, and then considered the word so defined to be a member of the same class.

Similarly, in the case of the second paradox, we defined a certain integer, N, by reference to the class of all numbers definable by less than thirty syllables and then considered the integer N, thus defined, to belong to the same class. This feature of defining some entity by reference to a class of entities and then allotting the defined entity to the same class is the pivot, not only of the two paradoxes I have described, but of each of the many variations which have been invented.

This feature common to all the paradoxes suggests a way of preventing their occurrence. We need only to discriminate between adjectives such as "long", "short", "English", "German", which explains properties definable without reference to adjectives, and adjectives *of* adjectives, whose properties can be defined only by reference to adjectives in the primary sense. Let us call them type one adjectives (primary sense) and type two adjectives (derived sense). Then type two adjectives can be attached only to type one adjectives; it will be nonsensical to ask of a type two adjective, such as "heterological", whether an adjective of type two applies

to it. Once this distinction is made it becomes impossible to formulate the fallacious argument embodied in the paradox and consequently no contradiction can arise. Similarly in the case of numbers definable by less than thirty syllables we should find it necessary to distinguish between type one definitions in which no reference is made to other definitions, and higher type definitions which involve reference to definitions of lower type.

This is the line of thought which Russell pursued in constructing his famous Theory of Types which was designed to eliminate the logical and mathematical paradoxes from the system of *Principia Mathematica*. Unfortunately, the solution was too drastic ; while successfully eliminating contradictions it simultaneously rendered invalid important types of argument of frequent occurrence in mathematics. For, if we draw a sharp distinction between the members of a class and objects defined by reference to the whole class, we shall find it very hard to preserve the generality of mathematical statements. And we shall need to pay very careful attention to the ways in which mathematical objects are defined. The proof of the existence of an infinite number of primes, for instance, is sometimes put in the following form : suppose $p_1, p_2, \dots, p_n$ are all the primes. We then prove that $p_1 p_2 \dots p_n + 1$ is either a prime which is greater than each of $p_1, p_2 \dots, p_n$ or is divisible by a prime which is greater than $p_1, p_2, \dots, p_n$, and thus obtain a *reductio ad absurdum* of our postulate of the existence of only a finite number of primes. This argument, however, now becomes invalid. $p_1 p_2 \dots p_n + 1$ is defined by reference to *all* the primes (compare the definition of N in our second paradox) and cannot therefore be regarded as a number of the same sort as $p_1, \dots, p_n$. In this particular case the difficulty thus raised can be circumvented, but no amount of ingenuity has been able to save certain important arguments used in the mathematical theory of functions. According to the Dedekind-Cantor of the real number, one of the most important links in the process of arithmetisation which I have previously described, a real number is defined in terms of a certain collection of rational numbers. We ought to call such numbers type one real numbers, and distinguish them sharply from real numbers of type two which are defined in terms of collections of real numbers of type one. Now in most existence theorems, for instance the theorem that a function continuous throughout an interval attains its upper bound at some point in the interval, we establish the existence of a real number by considering properties of *all* the real numbers in a certain class. If the new member whose existence is thus demonstrated could not be treated on a level with real numbers of lowest type (*e.g.* could not be compared with them in respect of magnitude), the whole point of the proof would disappear. We must not exaggerate the difficulties which thus arise. If we are examining a specific function, say the exponential or the ζ-function, we can often define particular values by means of a sequence of rational numbers and thus obtain real numbers of primary type ; it is when we come to general theorems

about all functions of a certain sort, however, that the theory of types cripples the mathematician, and threatens to excommunicate useful and elegant theories whose validity the practising mathematician can hardly find it in his heart to question.

The dilemma in which mathematical philosophy has found itself since the appearance of the logical contradictions has been misinterpreted by those writers who, like newspaper sellers gloating over the latest disaster, have made capital out of the so-called crisis in mathematics. The situation is not so much that mathematics itself is in danger : for no living science ever has been or ever will be completely consistent, and absence from internal contradictions has never been either a necessary or a sufficient condition for mathematical progress. But mathematics can never be satisfied with a state of affairs in which its own principles are unformulated : to such formulation the emergence of the logical paradoxes administered a severe check. They threatened to render abortive the movement towards the construction of the whole of mathematics as a unified deductive and abstract system. For it is essential to an abstract system that the rules of the game should be clearly and unambiguously stated ; unless we have such rules for the manipulation of symbols, constant reference will need to be made to the *meaning* of symbols and the abstract character of the system, with all its attendant advantages, will be lost. The rules implicit in everyday language are too lax and lead, as the paradoxes show, to contradictions ; but the rules proposed by Russell are so strict as to exclude important branches of pure mathematics. Yet the very nature of mathematical activity which has produced the whole movement towards an abstract system which I have described, demands rules of *some* sort.

Russell attempted to avoid the difficulties raised by his theory of types by introducing the so-called axiom of reducibility, of which the best that can charitably be said is that even if it were true it could never be *known* to be true. It has found little favour with subsequent writers of whom many have tended to follow Brouwer and Hilbert in rejecting the attempt to prove the identity of mathematics and logic. But closer examination of the schools of Formalism and Intuitionism, which seem to approach the whole subject from suppositions differing widely from those of Russell and Whitehead, shows that the quest for the abstract deductive system has by no means been abandoned. Indeed, each of these schools, in their own way, take the deductive ideal more seriously even than was the case in *Principia Mathematica*.

The most recent researches in this field have shown that the difficulties first revealed by the paradoxes are connected with certain basic and irremediable deficiencies in the deductive ideal. Given any mathematical system in the deductive form—and sufficiently complex to include the theory of the real number, we now know a uniform procedure for manufacturing theorems in the system in question, which can be seen to be true, but cannot be proved to be

true. Thus, in a certain well-defined sense, the deductive ideal cannot be completely realised: given any deductive system, no matter how extensive, we may always have occasion to appeal to principles which cannot be expressed in terms of the notation used in that system. Mathematics is perpetually unfinished, although we can never point to any particular department of it which cannot be reduced to order.

The recent history of the deductive ideal in mathematics is not quite as remote from the teaching of mathematics as it may at first sight appear. For the one general reflection that seems to emerge from this period of development is that the deductive ideal in mathematics is itself subject to change and evolution. The history of this period shows clearly how the notion of an abstract deductive system, the common meeting ground of mathematics and logic, which seems so simple and distinct in the case of a primitive algebra like Boole's, changes, becomes progressively more subtle, and drives mathematicians into making the most difficult and complicated re-arrangement of their own subject. The process of arithmetisation, arising largely from the technical needs of mathematics itself, issues in a critique of the very pattern of deductive form which it is its aim to reach. Progressively stricter requirements are made of mathematical arguments until the final ideal begins to appear like the end-point of an infinite series—no matter how much nearer we approach, we still remain at an infinite distance.

Recognition of this evolving, receding, nature of the deductive ideal frees us from the superstition of regarding the ultimate type of mathematical system as determined eternally, whether by Euclid, by Hilbert, or by God, and allows us to pay due attention to those non-formal, but no less important aspects of mathematical thought, which I have been compelled to neglect in this paper. Without in any way denying the supreme importance for mathematics of a deductive ideal we can recognise the difficulties which attend its pursuit, and adopt a conception of teaching in which the emphasis is laid, not upon the inaccessible ideal but upon the means of approximating to it, not upon the end product of the infinite series constituted by the evolution of mathematics, but upon the law of formation, the mathematical method.

M. Black.

1185. . . . that branch of arithmetic called practice. It propounds the gritty kind of sum which requires you to find the cost of, say, 267 articles at £3 8s. 9¼d. each (what articles, by the way, are likely to be sold at these odd prices? Perhaps combined topees and diving-helmets, or corduroy nosebags for elephants? . . .)—Mark Grossek, *First Movement*, p. 83.

[Per Mr. P. J. Harris.]

1186. Rule of three masqueraded as simple proportion, but it was the same stuff about cisterns being filled and walls being built and grass being eaten.—Mark Grossek, *First Movement*, p. 84. [Per Mr. P. J. Harris.]

TEACHING THE COMPLETE DUFFER.*

Mr. B. L. Gimson (Bedales): I have never taught a complete duffer; and I doubt whether any of you have either. But I daresay we should all agree that we have had pupils before us who at first sight seemed to be strong candidates for such a designation. Duffers they may have been, but it is the epithet "complete" that I cavil at, indicating as it does an abysmal stupidity which shuts out the faintest ray of hope of ever bettering their condition.

Our first problem in teaching duffers, it seems to me, is to convince ourselves *and them* that they are not hopeless. How often have we seen that glazed look come over their mulish faces, as much as to say: "What's the use of trying to teach me this stuff? I can't ever understand it." A deadlock is bound to ensue until we can find a way of establishing confidence.

Perhaps we expect too much of them. At sixteen we expect them to be able to manipulate fractions and keep the decimal point in the right place; and they (who have been trying for years to conceal a weakness in these tasks, due to a misspent youth) expect black looks or something worse when the weakness is dragged out into the daylight. I had a boy of seventeen once who was obviously frightened of some easy account keeping that I'd set him. And it presently appeared that he wasn't really sure of his adding. When he found that I wasn't going to "row" him, his relief was immense, and together we set about some daily drill which, after a term's steady plodding, brought him the confidence that he had never before enjoyed. He went into business after he left, and later he wrote telling me of the hours he had to spend checking other people's accounts. But he wasn't afraid of himself any longer, and now he occupies a responsible position in one of our biggest provision firms.

I don't believe the *complete* duffer in mathematics exists; just as I am sceptical of one who is said to have *no* ear for music. Everyone is interested in greater or less degree in expression through speech; and music and mathematics too may be regarded as forms of utterance quite as universal in their appeal. Not so highly developed, it is true, but latent; and I have failed to find the boy or girl without a glimmer of interest in the numerical and spatial relationships of common things in the world about him.

The teacher's business is to find out where that interest lies, to awaken it, to encourage it, to make it grow into something strong and abiding. How are we to discover the switch that shall flood the duffer's mind with light?

There are two motives that we may hopefully appeal to: Curiosity and Utility. It is no use expecting the logic of mathematics or the elegance of a demonstration to touch the imagination of the duffer. His mind probably never reaches the so-called Stage C of Geometry; its normal milieu is Stage A with an occasional excursion into Stage B.

* A discussion at the Annual Meeting of the Mathematical Association, 5th January, 1938.

His mind is practical, and he prefers to deal with concrete objects. So I put Curiosity and Utility as the two motives that may awaken his sleeping interest.

Curiosity first, because in the history of the race that surely came far earlier than utility. Man was asking the why and how of things long before he gained control over them for the purpose of earning his bread and butter. So let us put before the duffer anything that may stir his curiosity : magic squares, quaint properties of numbers, the sieve of Eratosthenes, dissection puzzles in equivalent areas, properties of geometrical figures by paper folding, missing digit problems, the " four fours ", and hosts of such puzzles which, for the value they possess in stimulating the duffer, are far less frivolous than they appear to be. I daresay my friend Mr. Boon may have some helpful things to say in this connection, so I will not enlarge on this aspect of the matter.

The history of mathematics may be a fruitful source of suggestion. What puzzled the race in its early stages will puzzle and intrigue our boys and girls. How does a sundial work? Why are there 360 degrees in a circle (except in Germany)? How was the length of a year determined, and how did the present subdivisions of time come to be settled? How did our numerals come to have their present shapes, and how did they improve over other systems of notation in the past? The use of symbols, and the extraordinary power they give to man—a fascinating study. The use of the abacus and other calculating machines as contrasted with methods of calculating on paper. The meanings of words used in mathematics, the names of our weights and measures, the names of geometrical figures, and how they came to be derived.

With one or other of these devices surely we can find a chink in the armour of the duffer and tickle him into willing response. I don't suggest for a moment that one should lay out courses in any of these subjects to be followed exhaustively, or that a large proportion of time should be spent on mathematical puzzles. But in the initial stages I should be willing to " waste " a whole term, if need be, if one were rewarded in the end by capturing the pupil's interest.

Sooner or later, however, he will want something more solid to work on. And here I should appeal unashamedly to utility motives. If the boy or girl knows what his later career is going to be, I should go for that for all it was worth. I should *not* try to turn him out as an expert bookkeeper, or a skilled architect or surveyor. But I *should* try and select problems that bore directly upon the job that he was interested in.

Unfortunately children more often than not are uncertain as to their future career, so then one has to appeal to utility on a less individualistic basis. Here the strong card is human interest. Again and again one hears the complaint that mathematics has nothing to do with real life. Alas, it is too often true ; but it need not be, in selecting work for duffers. Let us go first and foremost for problems of real life : account-keeping, especially the accounts that

show analysis of expenditure into parallel columns ; the Arithmetic of Citizenship with its familiar topics of local rates, the national budget ; saving in its various forms, from post office bank and savings certificates to building societies and the usual stocks and shares ; insurance both personal and national, life policies, householders' policies and so on ; there are problems for the householder concerning electricity, gas, the telephone, and the advantages and disadvantages of hire-purchase on the instalment plan. These and kindred topics one is glad to see nowadays included in school courses to an extent that was unthought of ten years ago. And there are now quite a number of useful little textbooks on these subjects available for the teacher, though anyone who has worked with duffers will realise the enormous advantage to be gained by using original material (rate demand notes, insurance policies, newspapers, etc.) rather than textbooks.

But we might, I think, explore further along these lines. There is much that can be done with house plans. Let your pupil study a house plan in detail, or, better still, try and design one himself. Let him criticise it from the point of view of convenience in running, suitability of aspect, economy of design in regard to cupboard space and so on. Let him consider the problem of furnishing a room, using current catalogues of prices. It is a fallacy to suppose such domestic details appeal only to girls. I have found the boys in a mixed class quite as keen to tackle their bit as the girls ; though if one apportions the various rooms among the class, it may well be that the girls will take the kitchen department, while the boys will be interested in living room or problems of heating and lighting.

Again, the current interest in national fitness can be made a starting point for various lines of study. The whole problem of food values, nutrition, suitable planning of meals both for private families and for institutions may occupy a girl for a term or more. Vital statistics are becoming everybody's concern, and (especially if records of height, weight, etc., are kept periodically at one's own school) much interesting work of a comparative character can be done on the growth of children ; charts being drawn to show the health of the community now as compared with a generation ago, or as compared with another section of the community living under different economic conditions. And this may well lead to a study of population statistics. The work of Professor Carr-Saunders, Dr. Kuczynski, Dr. Grace Leybourne and others provides ample material for an investigation on the trend of population ; and this suggests a wide field of research into the possible effect upon the various social services.

I have said enough to indicate the kinds of topic that one can discuss with duffers in mathematics. The suggestions made here are not purely fanciful. There is not a single one that I have not tried out on duffers in my own classes—and found to work. But there have been failures too—what is one man's meat is another one's poison.

May I conclude with a few remarks on the technique of making mathematical material attractive to duffers. As far as possible appeal to the eye: use diagrams, draw charts, pictures too (if you have the gift). You know how the advertisement with the right picture carries its message with greater force than any words. How many people read the letterpress of *Punch* for the hundreds who turn over its pages for the illustrations? What is history without maps? To most of us the graphical way is the vivid way.

Which brings me to graphs. I cannot understand why more attention is not paid to graphs in the normal school course; but certainly for the weaker brethren it is the most powerful method of bringing home valuable mathematical ideas. Without recourse to the study of algebraic graphs, simply confining oneself to statistical material, one can give children an insight into mathematical processes which they may never get purely through figures. The idea (*a*) of proportion through ready-reckoner graphs; (*b*) of rates, first through travel graphs and then with other straight or curved graphs in which the gradient may be studied; (*c*) of the whole principle of functionality, the distinction of the dependent and independent variables, and the difficulty of dealing with problems where the variables are many; (*d*) of positive and negative quantities; (*e*) of the need to watch the degree of accuracy of one's data, and the practical meaning of approximation; (*f*) of the possibility of interpolation, and the dangers of extrapolation or prophecy.

I believe graphs to be worth all the other methods put together, especially if livened up by the use of coloured chalks or inks. And with graphs we may couple other pictorial methods of exhibiting statistical material. You may perhaps have seen an interesting publication which came out a little over a year ago called *The Home Market*, in which a quantity of data useful to business men is presented not only in tabular form, but is made vivid by coloured silhouettes—rows of roguish boys, demure maidens, prosperous business men, old women bent with age, to illustrate distribution of population; and so on throughout the book. I suppose this sort of thing appeals to the business man for whom the book is designed; I *know* that it is exactly what my duffer wants to make the dead figures come alive.

Finally the teacher of the duffer must be prepared to go to infinite pains to find material suited to each individual duffer. Seldom can one take the class as a class. We must individualise the work. By Dalton assignments, or some similar system, we must try and fit the work to the pupil. Then each can work at his own pace at the work best suited to his mental capacity. My experience with mixed classes of boys and girls ranging from fifteen to eighteen is that I can keep about twenty happily occupied at the same time. Whether it is possible to deal with larger classes on this plan I can only leave to the courageous experiments of those teachers who unhappily have to cope with thirties and thirty-fives.

Finally, let us never despair of the duffer sitting before us. He

is not hopeless. With encouragement and infinite patience we can help him to master the little he does know. We can bring him the confidence he so sorely needs; so that in time he may find the courage to exercise an independent judgment in dealing with every-days problems of a mathematical character.

There is no such thing as a complete duffer.

Mr. F. C. Boon: The value of any contribution I can make to this discussion is subject to the limitations of my experience: I am no psychologist; I have never taught girls; and I don't remember having met the complete duffer. The duffer of my remarks is the backward boy we all know and have despaired of.

I have been told that there is only one thing to attempt with him—to cram him with a few simple mechanical processes. I regard that as a counsel of despair. Not only is cramming dangerous, but in my experience the duffer is as anxious to think things out and, if he is allowed to go at his own pace, as capable of arriving at a grasp of principles as the average boy.

I have also been told that he must not be put in a C class as that gives him a label of inferiority and induces an inferiority complex. Again I disagree. Label or no label he knows he is a duffer, and is content or even grateful to find himself in a class of kindred intelligences. To place him in an A class where he is hopelessly outpaced is to reduce him to despair.

My prescription as a general practitioner is: he must have a fair deal. In my early days he was allotted to an elderly gentleman of respectable classical attainments whose mathematics were characterised by lethargic ignorance. At the same time, in the same schools, brilliant young mathematical masters were engaged for a few potential mathematical scholars. I put it to you that mere justice demands that special arrangements should be made for the specially slow if they are made for the specially bright, and that our duty as teachers is as much concerned with enabling the duffer to get full value out of his school life as in helping the bright one to realise his potentialities.

The special arrangements include: classes as small and as homogeneous as staffing conditions permit, and the provision of a teacher who is interested in his subject and sympathetic to the duffer. It is of the utmost importance that the duffer should find himself in an atmosphere where he can give his silly answer or ask his silly question with no fear of the derision of quicker boys or the snub of the master. The silly question or answer is the clue to his difficulty.

The pace of the class must be slow. Dr. Ballard has said that "every child has his own tempo". The runner cracks if he is forced beyond his pace. A piece of machinery breaks down from being overdriven. The duffer when hurried becomes a straggler and drops out; while moving at his own pace he frequently displays unsuspected interest and intelligence.

In the duffers' class there must be more oral work, more repetition and more frequent revision than the average boy requires.

About 1924 or 1925 I had an exceptional opportunity of using this prescription over a period of twelve years. In a school where numbers ranged between 840 and 940, all but language and history specialists did mathematics. In each part of the school there were sufficient duffers to form special classes ; and there was an adequate staff of mathematicians.

For mathematical instruction the school was divided into four blocks : (i) a Junior school in which promotions were normally made in January and September ; (ii) a Lower Block consisting of the IVth forms of the Classical and Modern sides ; (iii) an Upper Block made up of the Vth forms of these sides ; (iv) a combined block of all the science and engineering sides. In the Senior school the promotions were annual, taking place in September.

Throughout the school the syllabus was so drawn up that it permitted time for masters to try experiments and indulge their pet ideas. In the duffers' classes even the modest syllabus assigned was not exacted. It was an instruction to the masters to aim at thoroughness.

In the Junior school there were no duffer classes. If a boy failed to make reasonable progress in one half-year, he repeated the work, usually in a parallel class, in the next half-year. If this rate of progress persisted, it was taken into account in placing him when he reached the Senior school.

In each Block in the Senior school there were duffer classes, including two for certificate candidates in the Upper Block. One of the latter was taken by the man I once alluded to here as Mr. Johnson, who has the gift of making bright boys work enthusiastically and duffers hopefully, who can laugh at a duffer's mistake without laughing at the duffer ; the other by myself. At the beginning of the year my class was warned that we should not endeavour to cover the syllabus. There was no cramming. The geometry lesson was devoted mainly to riders and in algebra there was constant reference to principles. Each week each of these classes worked through a paper set at a previous examination, and while getting a sort of weekly revision learned to realise the scoring value of good work on part of the syllabus.

In the first five years my class averaged 73 per cent. of passes and Mr. Johnson had few failures. My failures generally had a second shot from Mr. Johnson's class and reaped the benefit of not having been crammed for their first attempt. These results were unfortunately too good. The examining body was changed and my successes dropped to between 50 per cent. and 60 per cent., Mr. Johnson's remaining in the 90's.

But it was in the Science and Engineering Block that we found the hard cases. At each entrance examination there appeared sons of migratory parents, boys from overseas, and duffers who had prolonged their preparatory life beyond the normal age. Many of these in spite of hopeless performances in the examination were admitted for economic and sympathetic reasons, and, joined by the worst

samples from our own Junior school, formed a heavy sediment of high average age.

Here the duffer mentality was most apparent, and yet results were obtained. I remember examining the bottom class when it was being taken by the master who took the mathematical specialists. The boys' work on decimals would have been distinctly creditable coming from an average class of lower average age.

When I took this class I think the greatest difficulty I encountered was a tendency of the pupil to translate my phrasing, designed to save him from the usual pitfalls, into one of his own. In some cases he had come primed with cheap-jack formulas from his preparatory school, for example, "Change sides, change signs". But even the orthodox "Subtract the same number from each side" is not safe from his powers of misinterpretation. $3x=21$ becomes $x=18$ when he takes 3 from each side. The teacher's difficulty is to ensure that he and the boy give the same meaning to "take away", and indeed the mathematics lesson must be continually a lesson in English.

In these cases of misinterpretation, frequent repetition is needed. The light that dawns to-day has faded to-morrow. Concrete illustration often helps to clear the fog for the moment, but does not

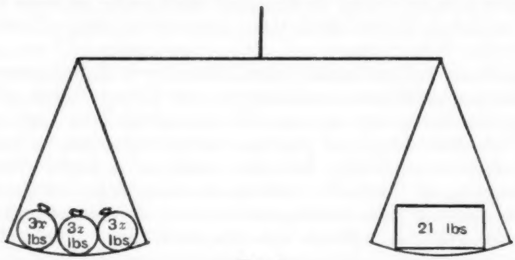


FIG. 1.

guarantee that it will not return. When the above equation is presented as in Fig. 1, after preliminary questioning of the meaning of $3x$, the absurdity of taking 3 from the left-hand side becomes apparent.

Once in my certificate set a boy solving $x^3=a^3b^3/c^3$ obtained correctly $x=ab/c$. In an average class that would probably have been allowed to pass. But with duffers one learns the wisdom of asking needless questions if only for the sake of repeated revision. When I asked how he got it, the answer was "By crossing out the cubes". This answer may interest those who cannot understand why sensible boys should proceed from $x^2=a^2+b^2$ to $x=a+b$.

Indeed, I may point out in passing, one gets many sidelights on the aberrations one encounters in the average class from one's experience with duffers. Only a few weeks ago in the algebra examination of an average boy in a certificate class I found $2\frac{5}{6}=2\frac{1}{3}x$ giving $\frac{5}{6}=\frac{1}{3}x$.

I found that many of the duffers take a delight in making cardboard models of solid figures and drawing patterns based on regular polygons ; in the latter the work was often imaginative and artistic. This undoubtedly helped the geometry. How vague the abstract may be to the duffer the following example will show. We were revising the theorem of the angle-sum of the triangle. The oldest boy in the class sprang to his feet and challenged me : " Can you show me those two right angles, sir? " I suspected him of having translated the enunciation into " There are two right angles in a triangle " and dealt with that, and then showed him the three angles together making two right angles by the well-known device of folding a paper triangle. This boy was one of my hardest cases—perhaps

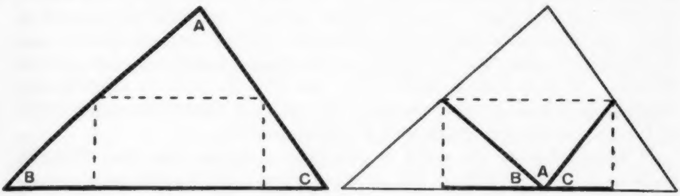


FIG. 2.

the nearest approach to the complete duffer I have met. But was he a duffer or had a succession of teachers failed to interest him? I once asked him what he was thinking of during his dreaming periods in class. He replied, " I only think of one thing, sir, horses."

For twelve years I spent nearly half my teaching periods with duffer classes. I refused to believe them incapable of a sound understanding of a certain amount of mathematics. I tried no heroic measures or freakish methods. I relied on patient explanation and reiteration (though by nature I am far from being patient with stupidity), and on keeping the pace of my teaching within the powers of the class. I do not record sensational successes. At times it was heavy going. But I think we gave our duffers a fair deal. My classes worked as cheerfully and as energetically as average classes. There were boys who stayed behind to tell me sheepishly that they had never thought mathematics could be so interesting. Once when a boy asked the not unusual question " What is the use of riders? " and I was about to explain that in his sense of the word there was none, a voice from the back interrupted " But they're good fun." And the interrupter was a very poor hand at riders. There were boys who had confessed themselves hopeless duffers who got a credit in their Certificate.

In conclusion I offer you my experience as a sort of personal recommendation. I found a duffers' class more interesting than a class of average boys. They showed a greater variety of mental and temperamental qualities. And I learnt from them a good deal

of what was useful in meeting the difficulties that arise in an average class.

Dr. F. H. Dodd: My work is mainly individual and with small numbers, and is approached essentially from the point of view of advising as a psychologist as to why a child is an apparent duffer. There is no such thing as a complete duffer. When I was asked to speak on this subject I questioned what was really meant by a duffer, and I defined it as a child who does not get on with mathematics and yet is not really backward. In times gone by teachers of mathematics used to take the line that there was a special ability for mathematics, and some children had it and some had not. Unfortunately of late people like Professor Spearman have shown us that the special ability for mathematics is a very small matter. In music, of course, and in art, it is a very big matter; the child backward in other ways may be brilliant in those subjects; but in classics and mathematics general intelligence is far more important than special ability. If you find a child who can get on well in classics and English in school work you can rely upon it that that child ought to be able to get on quite well in mathematics.

If you find that the child is generally a duffer, the first thing to do is to get it tested from the point of view of its intelligence. Even then you may find it is apparently backward, but an experienced observer can tell whether that is due to emotional problems or not. I have to approach the duffer from the point of view of diagnosis before treatment. I cannot say, " This child is a duffer ", and therefore apply such and such treatment, any more than I can say, " This person has a headache, and therefore the eyesight is wrong ", because the headache may be due to some other cause.

I think on the whole we may roughly divide duffers into two groups: those who are suffering from educational and environmental failure, and those who are suffering from some emotional disturbance. Educational failure is very important. For example, there is such a thing as bad teaching. I have been asked about children who are supposed to be going in for examinations, such as matriculation, and have been told quite seriously that they had been taught the formulae and rather discouraged than otherwise from discovering the formulae for themselves. They have come in some cases from schools where one would not expect it, yet the number I have seen does indicate that that method of approach to mathematics still exists.

Of course, one cannot help one big difficulty, and that is the breaking of the continuity of teaching when the child is moved about from school to school, or when, owing to illness, he is not able to get to school for a long period. In such cases the child loses links in the teaching.

There is also a very interesting thing to be noted, namely, the failure in mathematics of the child of high intelligence because the teacher does not realise that the child has such high intelligence. I have known a small child completely muddled as to mathematics.

She has come home and said, " I do not understand. What we are doing is to use straws, and we put 2 and 3 straws together and they make 5. But what do they use the straws for? " That child immediately imagines that there is some peculiar and mysterious reason for adding 2 and 3 together by means of straws. She said that they spent such a lot of time at school doing things with straws, and she wanted to know why the straws were used. It is not uncommon for a child to be muddled by being taught axioms as if they needed proof.

Again, the highly intelligent child may get bored. I once had to examine a boy who was at the bottom of the school, but when I tested him I found he ought to have been four forms higher; yet he never got a good report. Instead of giving his attention to school studies he spent his time with railway engine people. In his case it was sheer boredom. He did not see why he should waste his time repeating over and over again the same stuff, so he simply left it.

Then there is failure due to the general attitude towards mathematics that the subject is difficult, an attitude sometimes inspired at home and sometimes at school. A great many children get on quite well at home with their school work, but at school they hear a number of people talking of difficulties, and they imagine other difficulties and try to find them. There is also the difficulty of the child who feels that it is of no use trying to catch up with other people, or at any rate hoping to do as well as the grown-up people. I remember a child, very bored over mathematics, who came home one day and said she had a sum which was very difficult. I showed her how to tackle that type of sum, and she grasped the method and then took the sum and got it right. Next day she came back from school and said that everybody had got that sum wrong except herself, and that the teacher had done it on the board and had herself got the wrong answer. She had pointed this out to the teacher and the teacher had said, " Well, come and do it yourself ", and she did it. This child, having beaten the mistress on mathematics, went straight ahead.

Then there is the question of the child who does not like the teacher. We have always got to bear in mind that, this having happened, it does not matter at all how much the child likes the next teacher. I have to counsel what you may feel is an impossible thing, namely, that the duffer does need individual attention.

There is the other side of the question, that a child may not be suffering from educational failure but simply from emotional failure. It is an interesting fact that one finds that mathematics is the first thing to fail when the child is suffering from emotional conflicts. A general sense of inferiority may arise and persist right through school life or it may arise during adolescence. The commonest thing to happen to affect the work of such a child is for the child to retreat into a world of his own, in which he is not in the least interested in the real world round about him. This may be the type of child who

gets an excellent conduct report, but against every subject is the remark, "Could do better if he tried." That child may be quite intelligent, but he lives in a world of fantasy, cut off from the hardness of real life, and one has to tackle the child from that angle. If it is found that the child is regressing, if he is becoming a complete duffer in adolescence, it is either because he is shaping for some serious form of mental disorder, or, more likely, is retreating into an unreal world. I remember a boy whom I saw because he was doing badly in mathematics and classics. I tackled him from the mathematical point of view. I investigated his general interests. I discovered that he wanted to be an architect. That was a little suspicious. I went into it further, and discovered that he wanted to be an architect because he would like to build a Utopia of his own so as to carry on his daydream world. Well, in that case, one considered whether it was possible to associate mathematics with his Utopia. Among other things he was going to build a wonderful theatre. "Very well," I said, "you will have to consider the seating arrangements", and he immediately began to be interested in the arrangement of those hundreds of seats according to the best pattern, and that brought in mathematics. One has seen that sort of thing happen over and over again, and the complete duffer has been transformed into an exceedingly successful mathematician.

I heard only this morning about a boy who was a complete duffer at school until once he was absent for a few days. Then he reappeared and had got a scholarship in biology, which he had been teaching himself in his spare time and which was not a subject in the school.

I am not prepared to accept a duffer at his face value, nor am I prepared to accept one method of treatment. The first thing, as I see it, is this ; when one has to deal with the duffer one looks at him, and it is not difficult for trained observers such as we are to judge pretty well with what sort of child we are dealing, whether he is living in a world of his own, whether he is alert to the life around him or whether he is feeling inferior and nervous.

Then we have to go into the history, to ask questions about his academic career, and also his personal life. If the child seems to be backward in all subjects except perhaps music and art (we cut those out because they require specific intelligence), then we must discover the child's mental age. When we know that we know what to expect of the child. I have seen a serious nervous breakdown in a girl whose intelligence was say, 100, meaning by that the intelligence of the general population, but who was struggling to work level with girls whose intelligence was 110, which is sufficient to pass matriculation. Her "unconscious" knocked her out, and she could not go to school.

Then comes the testing of the child for the point of failure. It is no good trying to teach the duffer what he already knows, and therefore you must find out what he does not know and where his real difficulties come in. There are special tests for that, but the

ideal thing is for the person examining the child to make tests as he goes along. You must start by asking the child problems to which you are sure he can give the answer and insist on his giving all his reasoning aloud. Ask him, for example, to take 8 from 50, and when the child answers 42, that is not enough, for the essential thing is to learn how the child arrived at that result. If he tells you that he arrived at it by the following process, that 6 8's are 48, and 8 less than that is 40, but 50 is 2 more than 48, so that the result will be 42, you know that you have a child with a most complex method of thinking at the level of subtraction. You must concentrate on that point, and that means individual concentration because you cannot expect the whole class, even of eight or nine duffers, to have that same difficulty. When you have tackled one or two such points the child may be well on the road to become a mathematician and not a duffer.

Then you must see that the child has an incentive of interest. I am constantly coming across children who do not see the point of algebra. You can approach them by supposing that they are asked to organise a tennis tournament. They will have to consider how may sets there are going to be. Well, there at once is algebra. You get a very simple formula. Or again by the application of simple algebraic processes the child obtains rules which make certain arithmetic problems very simple. Success in this field gives the child a sense of power which is of great value, and it is very useful to bring home the value of algebra to the ordinary individual.

There is another example. The child very often has the idea that decimals are difficult. The other day I was coaching a child of twelve who thought decimals were difficult. I started off by talking about numbers, and we worked down until we wanted smaller fractions, one-tenth of one, and there I put a dot in front of the one to fix the units column, and presently as we went on the child had quite a shock to learn that it was so easy to do the addition and subtraction of "decimals!"

A great many children and all duffers have a dread of the terrible names that are given to some of these things.

The next point is that we need to make a friend of the child and so be able to understand and help him. In general, if you find that your duffer still remains apparently a duffer, after you have done all these things, you may be quite sure that there is something you have missed, a something which needs the use of special methods and knowledge. You may be dealing with a child who is really regressing mentally. But it is extraordinary how often you come across emotional breakdown which you can deal with quite easily if you only realise that it is not just a question of mathematics. No generalised method of teaching the duffer is going to be of much value. You have to adopt a diagnostic method of finding where the difficulty is, and then adopt methods of getting these particular things put right. When that happens, everything else falls into place, and the erstwhile duffer does his mathematics creditably.

GENERAL DISCUSSION.

Mr. C. T. Daltry (Roan School) said that he had been much interested in the point made by Dr. Dodd, that one should not try to teach the duffer what he already knew. In his own experience of teaching he had always found it a sound line to begin some quite fresh aspect of mathematics when meeting a slow form for the first time. Graphs were useful in this way, because even if pupils did know a little about them one could always lead on (or branch off!) to some interesting development. Another useful aspect was trigonometry. Both graphs and trigonometry were valuable because they merged together simple arithmetic, algebra and geometry. In teaching duffers he suggested that they should merge together the different elementary aspects of mathematics as much as possible. Trigonometry also encouraged real investigation. He remembered that two extremely slow boys were stimulated by that subject to borrow a theodolite and to measure various heights about the school. Their crowning achievement was the discovery that the minute hand of the school clock moved in jumps of $1\frac{3}{8}''$.

Often it was found that slow people had been kept at elementary arithmetic, for example, to the exclusion of more interesting work in algebra, geometry or trigonometry. That was a great mistake. One of the most useful functions of the mathematics master was to simplify the more advanced work, ruthlessly if necessary, so that the central ideas were brought down to the level of even the slowest person, *e.g.* graphs of functions should be chosen that required only the simplest calculations. The possible sources of failure in their teaching ought to be eliminated. In the matter of decimals he might suggest that in teaching the beginner it was well to restrict the instruction to tenths for quite a while, and only afterwards to go on to hundredths and thousandths.

His experience had been that the duffers did not always learn very much, but that their teachers learnt a great deal. The important thing was not what the duffers had done but what the teachers had done, and if the duffers only went away feeling that their time on mathematics had not been wasted and that they had been doing something interesting and significant, that was sufficient. He imagined that the chief function of the mathematics master in this world was, firstly, to make mathematics appear, as it was, interesting and significant, and, secondly, to create confidence among his pupils.

Mr. M. P. Meshenberg (Tiffins) said that he had the privilege of reporting the results of an experiment at his school with regard to a new scheme for teaching those whom they labelled " C " boys. The scheme was only two years old, but it had already been subjected to the test of external examination. After many years the powers that be had decided to try out a scheme of technical training in the C forms leading to the school certificate. This scheme included arithmetic, as the only compulsory branch of pure mathematics, and

geometrical and mechanical drawing. (These two were in London put together into the one subject.) At present the ordinary mathematics syllabus was supposed to be kept to in Form I and also in Form II, where, however, the Form II C boys did rather less algebra and geometry. But in the third form they ceased to be called Form " C " and became " Form Technical ", to express the fact that they were going to have a selection of subjects with the emphasis on their technical side, to drop algebra and geometry, and do geometrical drawing. (So far as mathematics was concerned the technical side was largely represented by flat-topped tables and drawing boards and mathematical models.) The master in charge took one form the year before last, and this last midsummer he presented the boys at the London General. These were boys who for three years had been in the school with no prospect of working for an examination, so that there had been the inevitable slackening of fibre and weakening of the quality of their work. But these boys, although they had, as he said, worked for three years with no examination in view, had achieved quite reasonable successes this last year in the examination. Their numbers had dwindled to 18. Whilst 11 reached the certificate standard in arithmetic, 15 of the 18 reached the certificate standard in geometrical and mechanical drawing. It must be borne in mind that the plane geometrical drawing involved quite difficult questions. One year it was required to draw a quadrilateral similar to a given one and one square inch more in area. Their technical brethren looked at such questions in a practical way, and the problems did not perhaps seem quite so difficult to them as they would appear to candidates in pure mathematics but the question had been set in one year in a Pass B.Sc. paper also.

Before this experiment took place he had himself had several experiences of teaching C forms, and he had been frequently tempted to the conviction that at least a certain number of boys in these forms might almost be candidates for the epithet " complete duffer ". He had found among these people the dread of long names which had been emphasised by a previous speaker in the discussion. In fact, his constant effort had had to be to devise a scheme of language, no matter how unorthodox, which should express the thing one wanted with the " minimum of syllables ". As an example, the word " equilateral " was something to be avoided as long as possible, also such a term as " congruent triangle ". The word " equidistant " was another great bugbear. In fact, these words should be postponed even with A and B boys. He had had a candidate who got his certificate about two years ago with distinction in each of the six subjects he presented. Yet this boy had refused to take mathematics in the sixth form. His story was interesting. He had startled his teacher back in II A with his extraordinary misuse of the word " equidistant ". " Why are these angles equal? " would get the answer " equidistant ". On tracing the matter back it was found that, unfortunately, in the form in which he had done his stage A geometry, great pains had been taken to make the boys

understand the meaning of every one of those big words—" rectangular parallelogram ", " rhombus " and so on. The word " locus " even had been dealt with. The boys had been made to understand those words so clearly that every one of them could get up and glibly repeat definitions of those words with examples. But when they reached the second year during which they were not going to be helped quite so much, hopeless confusion resulted in their minds. He had adopted the expedient of having some special short phraseograms made out in a list which he handed round.

He had found one feature among dull boys which another speaker had alluded to, but had not elaborated, namely, that most dull boys seemed extraordinarily keen to go on doing the things they found they could do, however dull those things might be. Thus he had often found them learn quickly and practise gladly the use of logarithms. He could not reconcile this with their general backwardness.

Mr. K. R. Imeson (Watford Grammar School) said that he would like to return to what had been said about the importance of presenting something fresh to " C " boys ; otherwise they might have a contempt for what they were doing, having done it so often before. This, of course, led to carelessness, which was one of the greatest difficulties they had to fight against with these duffers or " C " boys. One way in which they could be interested was to allow them to make discoveries for themselves. This could be done especially in geometry. He had sometimes found it a good idea, when doing riders, to let the boys draw an accurate figure, and from it discover certain results for themselves instead of always being told what had to be proved. Sometimes the discoveries they made would be wrong, and again it was useful to let them prove that the results were false. One specially useful figure in this connection (which the speaker illustrated on the blackboard) was that of two circles in contact, together with their common tangents and the semi-circle on the line joining the centres. Among other interesting facts which even a " C " boy would discover from an accurate figure was that the above semi-circle touches the direct common tangent of the two given circles. Quite a number of them would be able to prove that the general proposition was true. Several correct results were available for discovery from such a figure, and a number of incorrect ones also might be put forward, and the proof or disproof of the results obtained was a very useful exercise. (See Note 1288.)

Mr. Miller (Thorne) desired to make a point which had arisen in the morning in another discussion, namely, that of the abstract versus the concrete. The " C " boy demanded that his mathematics should be concrete and not abstract. It was not merely a question of subject-matter but also of language. To invite the " C " boy to contemplate a frustum of a cone did not help him very much, but if one asked him to think of an ordinary lampshade, or even the kind of thing elephants perched on in circuses, there was a very fair chance that he would know what one was talking about.

Talking of lampshades reminded Mr. Miller of an experience with a boy who had apparently done remarkably little work in his earlier career, and who was an excellent approximation to the " complete duffer ". This boy came one day with the request to know what shape the parchment cover of a lampshade ought to be. He was referred to the relevant chapter on the cone, with a passing reference to similar triangles, and left to his own devices. About a fortnight later he presented the results of his calculations for verification. Rather remarkably they were correct. Some time after this he came with the information that the lady who was the cause of this unwonted activity had made the lampshade successfully, and he then added the remark : " You know, sir, this is the first time I knew that mathematics ever did anything."

" C " boys were apt to grow up in the belief that mathematics never " did anything ", and the belief was by no means confined to them alone. Not unnaturally they were not prepared themselves to do anything with a subject which was apparently so devoid of connection with the world in which they lived. Teachers were apt to assume that their pupils were as familiar with the applications of mathematics in everyday life as they were themselves. The teacher, for example, might know that certain forms of sextant make use of the " angle at the centre " property, but to the pupil that fact was probably not so familiar. There was a temptation, especially under the shadow of impending examinations, to take the applications of mathematics for granted and to concentrate on the abstract mathematical principles to the exclusion of all reference to the everyday world. To the " C " boy such mathematics was largely meaningless, and it would not be unfair to suggest that the duffer was quite often the product of teaching which was far too abstract.

Miss D. J. Milner (Queen Mary High School, Liverpool) asked whether, in view of the large audience attracted by the subject and the interest taken in the discussion which was being cut short by reason of the time, this would not be a good subject for the next pamphlet to be issued by the Association. She thought that some of the members who were interested should get down to a certain amount of thorough research on this subject. It was one particularly in which a number of them might collaborate. The more we discovered about the proper way to handle and help boys and girls commonly called duffers, the more we should automatically learn about teaching the average pupil.

The Chairman (Professor E. H. Neville) said that the new Teaching Committee was meeting that day, and the suggestion made by the last speaker would be fresh in the minds of its members.

1187. Lagrange's reaction to descriptive geometry was like M. Jourdain's when he discovered that he had been talking prose all his life.—E. T. Bell, *Men of Mathematics*, pp. 214-215. Index, p. 648. Jourdain, P. E. B., pp. 214-215.

[Per Mr. G. N. Bates.]

MATHEMATICAL NOTES

1282. *A topological puzzle.*

Prove that it is impossible to join each of three points A, B, C to three others X, Y, Z in a plane by curves of which no two intersect, two of the six points, and only two, lying on each curve.

This theorem is proposed as a puzzle in Dudeney's *Amusements in Mathematics*, in the form that houses A, B, C are to have water, gas and electricity laid on by mains which do not cross. Dudeney says that the puzzle is an old one. Can any reader of the *Gazette* supply other references to this or kindred theorems?

No restriction is here imposed as to the positions of the six points A, B, C, X, Y, Z.

By way of proof, the following would, I think, be accepted by the majority of geometers.

Consider three pairs of curves from A to B. (i) AX, XB; (ii) AY, YB; (iii) AZ, ZB. Whatever the positions of A, B, X, Y, Z, they can be drawn not crossing one another. Each two of them, for example (i) and (ii), therefore enclose an area; and C and Z must lie either both inside or both outside this area; otherwise the curve CZ would have to cross one of the curves bounding the area.

Next consider two such areas, for example those contained by (i) and (ii) and by (i) and (iii) respectively. They have a common boundary (i), and the third area contained by (ii) and (iii) must be either their sum or their difference. Thus in all circumstances one of the three areas is the sum of the other two.

Suppose that the area $AXBYA$ contained by (i) and (ii) is the sum of the other two. Then AZ, ZB lie within it and divide it into two. Z lies inside $AXBYA$, and therefore C must do the same. But by the same reasoning, since X lies outside $AYBZA$, C must also lie outside it; and since Y lies outside $AXBZA$, C must also lie outside it. Unless C lies outside both the smaller areas and inside the larger area which is their sum—an obvious impossibility—one of the curves CX or CY or CZ *must cross one of the six named above.*

H. W. Richmond.

1283. *On relative velocity.*

Mr. Doughty's note on the above subject (No. 1250, *Mathematical Gazette*, October 1937, p. 285) drew my attention for the first time to Mr. Young's note (No. 1046, *M.G.*, Oct. 1932, p. 270).

For many years I used two notations which were very like those mentioned in the above notes: for the velocity of A relative to B sometimes I used $V_{A/B}$ and sometimes V_{A-B}. As all velocity is relative I never used Mr. Young's V_P; I should use V_{P-E}, E standing for the earth.

With the first notation $V_{A/B}+V_{B/C}=V_{A/C}\left(cf.\ \frac{A}{B}\times\frac{B}{C}=\frac{A}{C}\right)$, and with the second $V_{A-B}+V_{B-C}=V_{A-C}$ [*cf.* $(A-B)+(B-C)=(A-C)$].

Both notations I found effective in teaching boys to deal with

problems on relative velocity, but ultimately I came to the conclusion that V_{A-B} was the better notation.

In the early stages I avoid reducing one body to rest by adding on the reversed velocity; this helps the idea of adding vectors to come into greater prominence. My teaching practice seems to be closely allied to that of Mr. Doughty, but I prefer my method of showing what is given.

I may perhaps illustrate this more easily by a worked example from Siddons, Snell and Dockeray's *Mechanics* (p. 283), in which book the reader can find full particulars of the use of this notation.

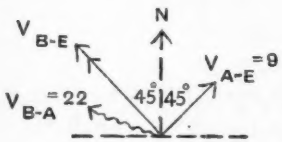
A bird flies at the rate of 22 *mi./hr. in still air. The wind is blowing from the S.W. at* 9 *mi./hr. In what direction must the bird try to fly in order that it may actually go N.W., and what will be its velocity in the N.W. direction?*

Take B for the bird, E for the earth, A for the air.

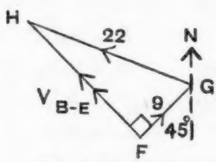
We are given vel.$_{A-E}=9$ mi./hr., N.E., vel.$_{B-E}$ is to be N.W., magnitude to be found, vel.$_{B-A}=22$ mi./hr., direction to be found.

Draw a freehand space-diagram showing the various velocities.

[In drawing this, note (i) that $v_{B-A}+v_{A-E}=v_{B-E}$, so v_{B-E} may be called the resultant velocity and so we mark it with a double arrowhead, (ii) that the direction of vel.$_{B-A}$ is roughly parallel to the line joining the other arrowheads, (iii) the arrow with the curly shaft indicating a velocity whose direction is not yet known.]



Space-diagram.



Velocity-diagram.

Then draw a freehand velocity-diagram FGH.

(First draw FG, then the direction of FH, and then cut off GH of the right length.)

Note that the arrows go the right way, for $\overrightarrow{FG}+\overrightarrow{GH}=\overrightarrow{FH}$.

The problem can be finished off graphically or by calculation.

A. W. Siddons.

1284. *A theorem of Newman and a theorem of Hurwitz.*

1. Newman † has shown that the number of matrices of given order satisfying the Clifford-Dirac equations

$$B_r^2=-I,\ B_rB_s=-B_sB_r\ (r\neq s)$$

is finite. If $2^q p$ (p being odd) is the order of the matrices, then their maximum number is $2q+1$, and further, if $\mathcal{R}$ of them have all their

† *Journal London Math. Soc.*, 7, pp. 93, 272.

elements real, and $\mathcal{I}$ of them have all their elements purely imaginary, then $\mathcal{R}-\mathcal{I}\equiv -1 \pmod 8$.

Hurwitz' theorem † states that the identities of which the simplest is

$$(x_0^2+x_1^2)(y_0^2+y_1^2)=(x_0y_0+x_1y_1)^2+(x_0y_1-x_1y_0)^2,$$

and which give the product of two sums, of n squares each, as the sum of n squares, only exist when $n=2, 4, 8$. We show that this follows at once from Newman's theorem. For if

$$(x_0^2+x_1^2+\ldots+x_n^2)(y_0^2+y_1^2+\ldots+y_n^2)=z_0^2+z_1^2+\ldots+z_n^2,$$

where
$$z_j=\sum_{i=0}^{n} a_{ij}y_i,$$

we can write, using matrices and vectors y, z,

$$z=Dy=(x_0D_0+x_1D_1+\ldots+x_nD_n)y$$

where the D_i are matrices with constant elements.

Equating coefficients of y_i^2, we have

$$\begin{aligned}(x_0^2+x_1^2+\ldots+x_n^2)I&=DD^*\\&=(x_0D_0+x_1D_1+\ldots+x_nD_n)(x_0D_0^*+x_1D_1^*+\ldots+x_nD_n^*),\end{aligned}$$

where the star denotes the transposed matrix.

Then $\quad D_iD_i^*=I.$

Hence $\quad D_i^*D_i=I. \quad (i=0, \ldots n).$

Let $\quad B_r=D_rD_0^* \quad (r=1, \ldots n).$

Then $\quad D_r=B_rD_0,\ D_r^*=D_0^*B_r^*.$

$$\begin{aligned}DD^*&=(x_0I+x_1B_1+\ldots+x_nB_n)D_0D_0^*(x_0I+x_1B_1^*+\ldots+x_nB_n^*)\\&=(x_0I+x_1B_1+\ldots+x_nB_n)(x_0I+x_1B_1^*+\ldots+x_nB_n^*).\end{aligned}$$

As this is equal to $(x_0^2+x_1^2+\ldots+x_n^2)I$, we have

$$B_r+B_r^*=0\ ;\ B_rB_s^*+B_s^*B_r=0 \quad (r, s=1, \ldots n\ ;\ r\neq s).$$

Since $\quad B_r=D_rD_0^*$, therefore $B_r^*=D_0D_r^*$,

$$B_rB_r^*=D_rD_0^*D_0D_r^*=I.$$

Hence $\quad B_r^2=-I,\ B_rB_s+B_sB_r=0 \quad (r\neq s).$

These matrices are of order $n+1$, and their number is n.

Hence by Newman's theorem, if $n+1=2^qp$, then

$$2^qp-1\leqslant 2q+1.$$

Since p is odd, the only numbers satisfying this are

$$p=1,\ q=0, 1, 2, 3.$$

Hence the only possible values of n are 1, 3, 7, and these only if we can find matrices of real elements which do not "interfere"; that is, if, say, B_r has a non-zero element in the (ij) place, the elements of all the other B in the (ij) places must be zero.

† *Werke*, II, p. 565.

2. Eddington gave the following example of five matrices (the maximum number) of order four, satisfying the conditions.

$$
\underset{a_1}{\left.\begin{array}{cc|cc} \cdot & i & & \\ i & \cdot & & \\ \hline & & \cdot & i \\ & & i & \cdot \end{array}\right.}
\quad
\underset{a_2}{\left.\begin{array}{cc|cc} i & \cdot & & \\ \cdot & -i & & \\ \hline & & i & \cdot \\ & & \cdot & -i \end{array}\right.}
\quad
\underset{a_3}{\left.\begin{array}{cc|cc} & & & i \\ & & -i & \\ \hline & -i & & \\ i & & & \end{array}\right.}
\quad
\underset{a_4}{\left.\begin{array}{cc|cc} \cdot & + & & \\ - & \cdot & & \\ \hline & & \cdot & - \\ & & + & \cdot \end{array}\right.}
\quad
\underset{a_5}{\left.\begin{array}{cc|cc} & & \cdot & + \\ & & - & \cdot \\ \hline \cdot & + & & \\ - & \cdot & & \end{array}\right.}
$$

where $+$, $-$ stand for $+1$, -1.

Newman pointed out that a real maximum eight-rowed set is obtained from these by forming the matrices

$$
\begin{array}{c|c} a_1a_2 & 0 \\ \hline 0 & -a_1a_2 \end{array}
\qquad
\begin{array}{c|c} a_1a_3 & 0 \\ \hline 0 & -a_1a_3 \end{array}
\qquad
\begin{array}{c|c} a_2a_3 & 0 \\ \hline 0 & -a_2a_3 \end{array}
\qquad
\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}
$$

$$
\begin{array}{c|c} 0 & a_4 \\ \hline a_4 & 0 \end{array}
\qquad
\begin{array}{c|c} 0 & a_4a_5 \\ \hline a_4a_5 & 0 \end{array}
\qquad
\begin{array}{c|c} 0 & a_5 \\ \hline a_5 & 0 \end{array}
$$

It will be found that these matrices do not interfere, and they lead to the following solution of the eight-square problem :

$$
\begin{array}{cccc|cccc}
0 & 1 & 2 & 3' & 4' & 5 & 6' & 7 \\
1' & 0 & 3' & 2' & 5' & 4' & 7' & 6' \\
2' & 3 & 0 & 1 & 6 & 7 & 4' & 5' \\
3 & 2 & 1' & 0 & 7' & 6 & 5 & 4' \\
\hline
4 & 5 & 6' & 7 & 0 & 1' & 2' & 3 \\
5' & 4 & 7' & 6' & 1 & 0 & 3 & 2 \\
6 & 7 & 4 & 5' & 2 & 3' & 0 & 1' \\
7' & 6 & 5 & 4 & 3' & 2' & 1 & 0
\end{array}
$$

This table is to be interpreted as follows. We have written 0, 1, 2, ... for $x_0, x_1, x_2, \ldots$ and $1'$, $2'$ ... for $-x_1$, $-x_2, \ldots$. Then the first and second rows give

$$z_0 = x_0y_0 + x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4 + x_5y_5 - x_6y_6 + x_7y_7,$$

$$z_1 = -x_1y_0 + x_0y_1 - x_3y_2 - x_2y_3 - x_5y_4 - x_4y_5 - x_7y_6 - x_6y_7$$

and so on. H. G. FORDER.

1285. *A property of the triangle.*

The theorem considered in note 1236 (*Gazette*, 1937, p. 158) has already been given, with some additional properties, in the " Note sur la Géométrie du triangle " by Neuberg in the *Traité de Géométrie* by Rouché and de Comberousse (see f. i., edition 1929, v. I, p. 505).

If A', B', C' divide the sides of the triangle ABC in the same ratio :

(i) The triangles ABC, $A'B'C'$ have the same centre of gravity.

(ii) They have the same Brocard angle.

(iii) AA', BB', CC' are equipollent to the sides of a triangle $A''B''C''$ having the same Brocard angle as ABC.

(iv) If a, b, c, S and a'', b'', c'', S'' are the sides and the areas of the triangles ABC and $A''B''C''$,

$$\frac{a''^2+b''^2+c''^2}{a^2+b^2+c^2}=\left\{\frac{a''^4+b''^4+c''^4}{a^4+b^4+c^4}\right\}^{\frac{1}{2}}=\frac{S''}{S}.$$

Property (ii) is equivalent to the theorem of note 1236, since the cotangent of the Brocard angle of a triangle having sides a, b, c and area S is $(a^2+b^2+c^2)/4S$. R. GOORMAGHTIGH.

1286. *Triangle properties.*

If A_1, B_1, C_1 divide the sides of $\triangle ABC$ in the ratio $m:n$ and AA_1, BB_1, CC_1 form the $\triangle PQR$, then

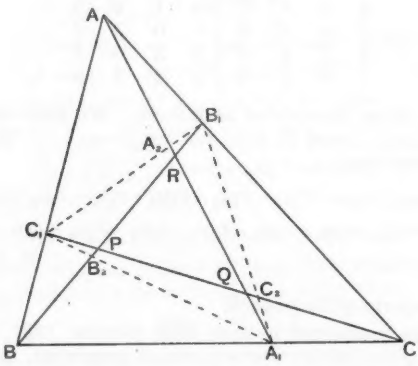
(1) all corresponding lines are divided proportionately,

(2) all corresponding areas are equal.

$$\frac{AQ}{AR}=\frac{m}{n}, \quad \frac{A_1R}{A_1Q}=\frac{m^2}{n^2}, \quad \frac{\triangle PB_1C}{\triangle PBC_1}=\frac{m^3}{n^3},$$

and if B_1C_1 cuts AA_1 at A_2, $\dfrac{A_2Q}{A_2R}=\dfrac{m^4}{n^4}$.

In fact A_2 and the corresponding points B_2 and C_2 divide the sides of $A_1B_1C_1$ in ratio $m^2:n^2$ and (1) and (2) still hold.



Similarly, if B_2C_2 cuts AA_1 at A_3, A_3 and the corresponding points B_3 and C_3 divide the sides of $A_2B_2C_2$ in ratio $m^4:n^4$ and so on indefinitely.

We may also note that $\triangle A_1B_1C_1=\dfrac{m^3+n^3}{(m+n)^3}\triangle ABC$

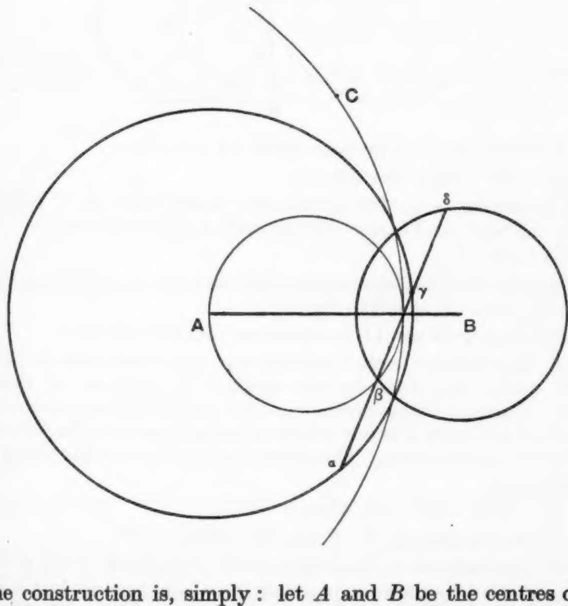
and
$$\triangle PQR = \frac{(m-n)^3}{m^3-n^3}\triangle ABC$$
$$= \frac{(m^2-n^2)^3}{m^6-n^6}\triangle A_1B_1C_1, \text{ etc.}$$

also if PA is joined
$$\triangle PAC : \triangle PBC : \triangle PBA = m^2 : mn : n^2.$$

C. H. HARDINGHAM.

1287. *A geometrical problem.*

Given two intersecting circles, construct a line which shall be trisected by its four intersections with the circles, these intersections occurring alternately.



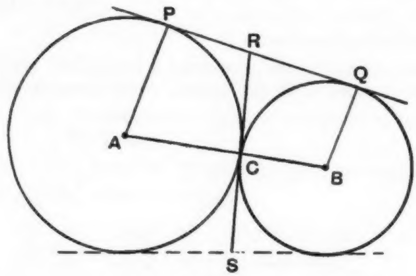
The construction is, simply: let A and B be the centres of the given circles. Let C be the vertex of the equilateral triangle on AB as base. Then the circle through C and the intersections of the given circles will cut AB in the same point as the required line, which can then be easily constructed.

I have proved the accuracy of this construction by applying conics and involutions, but should be glad to know if a proof by more elementary geometrical methods is obtainable.

W. J. HODGETTS.

1288. *Rider work in geometry.*

With reference to the article on "Rider Work in Geometry" in the *Gazette* of May 1936, in which examples are given of facts which may be "discovered" by a class before being proved, I have found the accompanying figure most fruitful in discoveries of this nature. It is the well-known figure obtained by constructing the direct common tangents to two circles, in the particular case when the circles touch one another.



The following discoveries were made by a C form.

(1) $CR = RP = RQ$; this led to:

(2) The semicircle on PQ as diameter passes through C and also touches AB at C, and hence PQ subtends a right angle at C.

(3) $RS = PQ$.

And lastly, the most interesting discovery of all, which can easily be spotted from an accurate figure:

(4) The semicircle on AB as diameter touches PQ at R.

A few false results were obtained too, and these had to be disproved, which was done by the method of analysis, or indirect method. Starting by supposing that the result is true, known facts are used to arrive at a result which either is known to be false or is true only in special cases. The following results were suggested and investigated:

(1) $AP = PQ$ (true only when $a = 4b$).

(2) QB passes through S (true only when $a = 3b$).

(3) QB is a tangent to the larger circle (true only when $a = 4b$).

Some of the boys' figures justified these conclusions, but a show of hands convinced the class that these facts were probably not true in general.

At this stage a discussion of the methods of analysis and synthesis, used by geometers since ancient times, might be given to an A form, and it might be explained that the method of analysis does not prove the correctness of a proposition unless the steps are reversible, while a fallacy can always be detected by this method.

K. R. Imeson.

1289. *Geometric figures with integral sides for illustrating Apollonius' Theorem.*

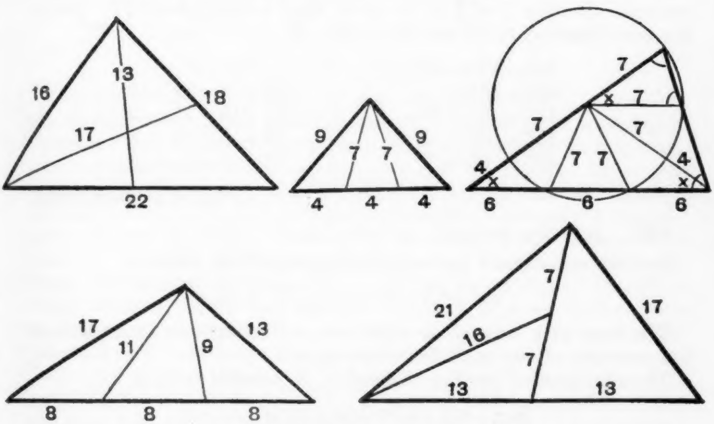
In teaching the theorem of Pythagoras and its extensions, numerical examples help the boys to appreciate their uses better, and if the first few examples come out in whole numbers the boys are not confused by decimals while tackling new ideas in geometry. There are well-known formulae for Pythagorean numbers, which enable one to get as many examples as one could wish for, for these theorems.

For the theorem of Apollonius, one can take any right-angled triangle obtained by the usual formula for Pythagorean numbers and draw the median from the right angle to the hypotenuse, but this may lead the pupils to think that the theorem only applies to right-angled triangles. What one wants is a formula of the type

$$s^2+t^2=2l^2+2m^2,$$

and each solution will give two triangles, since either l or m may be the median or half-base.

Incidentally one must test to see that the figure is possible.



Not knowing if a formula for Apollonian numbers had been made, and being in a hurry, I devised several, of which only one was of much use, though two others were of interest (most formulae give flat lines instead of triangles) :

$$(4a+b)^2+(2a+3b)^2=2(3a+b)^2+2(a+2b)^2\,;$$
$$(a^2-b^2)^2+(a^2+7b^2)^2=2(a^2-5b^2)^2+2(4ab)^2\,;$$
$$(10a^2-10b^2)^2+(12ab)^2=2(7a^2-b^2)^2+2(7b^2-a^2)^2.$$

Only the first proved of real use. By noticing if one triangle turned up twice, or if one half-triangle was isosceles or occurred in different cases, I have also been able to construct compounded figures.

Here is an examination question showing how use may be made of these results :

The longest side of a triangle of sides 13, 17 and 24 cms. is trisected; and lines drawn from the points of trisection to the opposite vertex. Find their lengths. (Hint : use the theorem of Apollonius.)

I should be most grateful to know of work on these lines, since mathematics is not my main subject, but only a side-line.

C. DUDLEY LANGFORD.

1290. *Note on a curious number property.*

While actually looking for something quite different some years ago, I noticed a curious property of numbers one less than multiples of 24, that may be of interest and even of use for a few older boys to try to prove. If any number one less than a multiple of 24 is split into two factors, then the sum of those factors is a multiple of 24. Conversely : if two numbers, neither of which are multiples of either 2 or 3, have their sum a multiple of 24, then their product is one less than a multiple of 24. This converse is quite easily deduced from the fact that the square of any number that is not a multiple of either 2 or 3 is one more than a multiple of 24. About the best illustration I know of is 455.

$$\begin{array}{llll} 455 = (24 \times 19) - 1, & & & \\ 455 = 455 \times \ 1, & & & 455 + \ 1 = 24 \times 19, \\ 455 = \ 35 \times 13, & & & 35 + 13 = 24 \times \ 2, \\ 455 = \ 91 \times \ 5, & & & 91 + \ 5 = 24 \times \ 4, \\ 455 = \ 65 \times \ 7. & & & 65 + \ 7 = 24 \times \ 3. \end{array}$$

C. DUDLEY LANGFORD.

1291. *Area of a triangle.*

Let the rectangular cartesian equations of the sides be

$$a_r x + b_r y + c_r = 0 \quad (r = 1, 2, 3).$$

The area of a triangle is half the product of two altitudes and the cosecant of the angle between them.

The equation of the line through 1, 2, parallel to 3, is

$$(a_1 x + b_1 y + c_1) + \lambda(a_2 x + b_2 y + c_2) = 0,$$

where
$$(a_1 + \lambda a_2)/a_3 = (b_1 + \lambda b_2)/b_3,$$

that is, $\lambda = C_2/C_1$, where capital letters denote the minors of small ones in the determinant $(a_1 b_2 c_3)$ or Δ.

Hence the above equation becomes

$$(a_1 x + b_1 y + c_1) C_1 + (a_2 x + b_2 y + c_2) C_2 = 0,$$

i.e. $$(a_1 C_1 + a_2 C_2) x + (b_1 C_1 + b_2 C_2) y + (c_1 C_1 + c_2 C_2) = 0,$$

i.e. $$-a_3 C_3 x - b_3 C_3 y + \Delta - c_3 C_3 = 0,$$

i.e. $$(a_3 x + b_3 y + c_3) - \Delta / C_3 = 0.$$

Thus one altitude is $\Delta / C_3 \surd(a_3^2 + b_3^2)$.

Similarly another is $\Delta / C_2 \surd(a_2^2 + b_2^2)$.

Hence the area of the triangle is

$$\frac{\Delta^2}{2C_2C_3 \surd(a_2^2+b_2^2) \surd(a_3^2+b_3^2)} \cdot \frac{\surd(a_2^2+b_2^2) \surd(a_3^2+b_3^2)}{a_2b_3 - a_3b_2},$$

or

$$\Delta^2/2C_1C_2C_3.$$

N. M. Gibbins.

1292. *Morley's triangle.*

With reference to Mr. Dobbs' article in the February *Gazette*, the extension to the other triangles by an elementary method was made by F. G. Taylor of Nottingham University College and myself and published in the *Proc. Edin. Math. Soc.* (14th November, 1913) under the title "The six trisectors of each of the angles of a triangle."

W. L. Marr.

CORRESPONDENCE.

THE TEACHING OF GEOMETRY IN VICTORIA.

To the Editor of the *Mathematical Gazette*.

Dear Sir,—The article on the teaching of Geometry in the May *Gazette* makes me think that the history of geometrical teaching in Victoria deserves being put on record before it is entirely forgotten. At some time early in the seventies of last century the Melbourne University adopted the principle, for its Matriculation examination, that any proof of Euclid's propositions would be accepted if Euclid's order was not violated. About 1879 Professor H. M. Andrew, Professor of Natural Philosophy at Melbourne University, and Mr. Pirani, lecturer in Mathematics, published an edition of Euclid in which modern proofs and modern methods of presentation were adopted in very many of the propositions. I was at school at the time and well remember the immense relief we got by exchanging Todhunter for this book. It consisted at first of Euclid I and II only, the third book being added a year or two later, after Pirani had been killed in an unfortunate accident. Book III was altered still more extensively than Books I and II. Books IV-VI were never written, probably because they were not then required for pass matriculation.

This edition of Euclid had a very great and beneficial effect on the teaching of geometry in Victoria and was almost the only book used in Victorian schools for pass geometry until many years later. I expect that Andrew and Pirani's work was due to the influence of the A.I.G.T.

R. J. A. Barnard,
Vice-President of the Victoria Branch of the Mathematical Association.

REVIEWS.

Über einige neuere Fortschritte der additiven Zahlentheorie. By E. LANDAU. Pp. 94. 6s. 1937. Cambridge Tracts in Mathematics and Mathematical Physics, 35. (Cambridge)

This tract, which is a development of lectures delivered by Professor Landau in Cambridge in 1935 at the invitation of the Faculty Board of Mathematics, gives a connected account of some of the remarkable advances which have been made in the additive theory of numbers since 1930. The lectures were in English, but the tract is in German and enjoys the distinction of being the first of the Cambridge tracts to be issued in a language other than English.

The subject matter of the tract centres round three famous conjectures of Waring, Goldbach, and Gauss, and will be best appreciated by a reader familiar with the chapters on Waring's problem, Goldbach's problem, and the theory of binary quadratic forms in the author's *Vorlesungen über Zahlentheorie.* The tract is, however, so written that only the classical elements of the theory of numbers are assumed. The key references to original sources are given in the Introduction; from these the reader can trace back the detailed history of the various topics for himself.

In Waring's problem the fundamental number is denoted, after Hardy and Littlewood, by $G(k)$. This is the smallest number s such that every sufficiently large positive integer can be expressed as a sum of s positive or zero kth powers. Its existence for each positive integer k was conjectured by Waring in 1770 and proved by Hilbert in 1909. But the first explicit estimate $G(k) \leqslant \bar{G}(k)$ in terms of an elementary function $\bar{G}(k)$ was found by Hardy and Littlewood with the help of their powerful analytical method, which reduces the study of an arithmetical function $r(n)$ (in this case the number of representations of n as a sum of s non-negative kth powers) to the study of the "generating function" $\Sigma r(n) x^n$ near its circle of convergence. The best estimate $\bar{G}(k)$ ultimately obtained by Hardy and Littlewood was exponentially large for large k. In 1934 Vinogradoff proved that $G(k)=O(k \log k)$ when $k \to \infty$. The basis of this remarkable advance is an elementary lemma giving (in suitable circumstances) a non-trivial estimate of a finite trigonometrical sum of the type

$$\sum_{m,n} e^{mn\theta i} \quad (\theta \text{ real}),$$

in which m and n range independently over arbitrary sets of integers (not necessarily consecutive). The ingenious method by which double sums of this type are made to play a part in Waring's problem is expounded, with notable simplifications due to Heilbronn, in Chapter 1. Whether $G(k)$ is actually of order k, as Hardy and Littlewood conjectured in 1925, is still undecided.

Chapter 2 contains a simplified proof of Schnirelmann's theorem (1930) that every integer $n > 1$ can be expressed as a sum of not more than A primes, where A is an absolute constant (independent of n). This is a step in the direction of Goldbach's conjecture of 1742 that every even integer can be expressed as a sum of two primes. Intermediate between these two assertions stands the result to which Hardy and Littlewood were led by the application of their analytical method, that every sufficiently large odd integer can be expressed as a sum of three primes. But this was obtained by them only on the assumption of an unproved hypothesis concerning the zeros of Dirichlet's L-functions. Schnirelmann's theorem was the first of its kind to be rigorously proved.

The proof of this theorem introduced the notion of "density", and suggested a deeper study of this notion for its own sake. The density α of a set

$\mathcal{A}$ of (distinct) positive integers a is defined as the lower bound of $A(x)/x$ for $x=1, 2, 3, \ldots$, where $A(x)$ is the number of a not exceeding x. The sum $\mathcal{S}=\mathcal{A}+\mathcal{B}$ of two sets $\mathcal{A}$ and $\mathcal{B}$ is defined as the set of distinct integers of the form a, b, or $a+b$, where a and b are typical members of $\mathcal{A}$ and $\mathcal{B}$ respectively. What can we say about the density σ of $\mathcal{S}$ in terms of the densities α, β of $\mathcal{A}$, $\mathcal{B}$? The proof of Schnirelmann's theorem uses two simple general propositions: (i) $1-\sigma \leqslant (1-\alpha)(1-\beta)$, that is, $\sigma \geqslant \alpha+\beta-\alpha\beta$; (ii) if $\alpha+\beta \geqslant 1$, then $\sigma=1$ (so that $\mathcal{S}$ includes all positive integers). It has been conjectured that $\sigma \geqslant \alpha+\beta$ when $\alpha+\beta < 1$. This is an unsolved problem, but important contributions to it were made by Khintchine in 1932 and by Besicovitch in 1935, and are described in Chapter 4. Khintchine's theorem is that the conjecture is true when $\alpha=\beta$. The proof of this is described by Professor Landau as "elementar und doch ein sehr kompliziertes grosses Kunstwerk". Chapter 3 deals with some special cases in which $\sigma > \alpha+\beta$. The sets

$(\mathcal{P})$	$1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots,$	
$(\mathcal{A}_k)$	$1^k, 2^k, 3^k, 4^k, \ldots$	$(k \geqslant 1),$
$(\mathcal{G}_a)$	$1, a, a^2, a^3, a^4, \ldots$	$(a > 1),$

all have density zero, but $\mathcal{P}+\mathcal{P}$, $\mathcal{P}+\mathcal{A}_k$, $\mathcal{P}+\mathcal{G}_a$ all have positive density. The first of these results was the crucial step in the proof of Schnirelmann's theorem, and the other two were proved by Romanoff in 1934. The proofs are "elementary", but employ the elaborate form of the "sieve of Eratosthenes" devised by Viggo Brun in 1920. There is a further group of theorems, introduced by Khintchine in 1933 and extended by Erdös, to the effect that, for certain special $\mathcal{B}$ with $\beta=0$ and *any* $\mathcal{A}$ with $0 < \alpha < 1$, we have $\sigma > \alpha$ and indeed $\sigma \geqslant \alpha+B\alpha(1-\alpha)$ with a positive B depending only on $\mathcal{B}$. Examples of such sets $\mathcal{B}$ are $\mathcal{A}_k$ and $\mathcal{P}$.

An appendix is devoted to a proof of Siegel's theorem on the class-number $h(d)$ of binary quadratic forms of negative discriminant d. A conjecture of Gauss (1801) suggests that $h(d) \to \infty$ when the discriminant d tends to $-\infty$. In 1934 Heilbronn succeeded in proving this by bridging the gap between results of Hecke on the one hand and of Deuring and Mordell on the other which showed that the conclusion followed from hypotheses of opposite kinds concerning the zeros of Dirichlet's L-functions and of the Riemann zeta-function respectively. In 1935 Siegel adapted Heilbronn's method to the proof of a more general theorem, which in the case of negative discriminants asserts that $h(d) > C_\epsilon \,|\, d \,|^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$ and some positive C_ϵ depending only on ϵ.

The subject is still advancing, and important new results have been published since the tract appeared. Thus Davenport and Heilbronn have shown that the set $\mathcal{S}=\mathcal{P}+\mathcal{A}_k$, proved by Romanoff to be of positive density, includes "almost all" positive integers, that is to say that $S(x)/x \to 1$ when $x \to \infty$. But the crowning event is Vinogradoff's proof of the theorem (obtained by Hardy and Littlewood only on the assumption of an unproved hypothesis) that every sufficiently large odd integer can be expressed as a sum of three primes. This theorem, if it had been available, might have found a very natural place in the tract; for the result supersedes Schnirelmann's theorem, and the proof combines in a remarkable way results and methods from various parts of the theory. The analytical method of Hardy and Littlewood, a novel adaptation of the "sieve of Eratosthenes", Vinogradoff's lemma on trigonometrical sums, and Siegel's theorem on the class-number are all brought into play. If it was a source of disappointment to Professor Landau that he should have concluded his tract too early to include a reference to this sensational discovery, he may well have derived consolation from the knowledge of the

part which he himself has played, by his own writings and by personal encouragement given to others, in stimulating the intense activity to which his tract bears witness. The news of his death on February 19 comes as a great shock to workers in the theory of numbers, who will appreciate all the more keenly the magnitude of the debt which they owe to him—an indebtedness greatly increased by a publication which makes available such a wealth of material in such a compact and readable form.

A. E. I.

The Axiomatic Method in Biology. By J. H. Woodger. Pp. x, 174. 12s. 6d. 1937. (Cambridge)

Woodger's book is, as he legitimately claims, an entirely original experiment. It is the first attempt to put some aspects of biological theory into axiomatic form. The main system which is proposed and worked out in some detail involves ten undefined signs. Of these two, P meaning " part of ", and T " before in Time ", are general concepts, while the other eight are more specifically biological; *org* for " organised unities ", U for " division or fusion ", *cell* for " cells ", m and f for " male and female ", *wh* for " whole organisms ", *Env* for the " environment " and *genet* for the class of " genetic properties ". By manipulating these few concepts, Woodger builds up more complex notions which suffice to express some biological laws, particularly those dealing with heredity.

From the logistic point of view, probably the most interesting feature of the system is the extensive use made of the class of relations known as hierarchies, which are defined as follows :

$$\text{hier} =_{Df} \hat{R}(R\epsilon \text{ as} \frown 1 \rightarrow \text{cls. } E!\, B\text{`}R \,.\, \text{C`}R = \overleftarrow{R}_{po}\text{`}B\text{`}R).$$

These are clearly branching systems, and in the biological examples the two most important cases are when $R\epsilon\, 1 \rightarrow 2$ as for cells, or $R\epsilon\, 1 \rightarrow \text{cls.}$ for sexually reproducing organisms. From the first case it is simple to define biologically appropriate concepts, such as that of the level (generation) of a cell in a hierarchy derived from a single beginner cell. The consequences of the second relation have not been so fully worked out, but look as though they would provide some interesting possibilities which might be useful, for instance, in the theory of inbreeding.

The main importance of the book, however, is as a contribution to biological theory, and it is from this point of view that it must be judged. Standing as it does completely alone as the first contribution to axiomatic biology, its value can probably not yet be correctly assessed. It has clearly the status of a forerunner, but its ultimate importance will depend on whether the method which it introduces can be developed into a valuable part of biology. There is no doubt that the particular fields which Woodger has chosen to expound are the most suitable for this type of approach. The elementary rules of heredity have the simplicity and clarity of a theorem of mathematics, but they are almost the only part of biology of which this is true. Even here it is only some aspects of the behaviour of the fundamental entities, genes, chromosomes, etc., which are simple and clearcut. The axiomatic approach quite rightly begins by cloaking the difficulties under its undefined symbols; for instance, the whole problem of biological organisation lies beneath Woodger's symbols *org* and *wh*, the question of the relation between genes as counters handed from generation to generation and as determinants of the course of development is implicit in the concept *genet*. These difficulties, however, are the real test, and the axiomatic method will not be of major importance unless it can say something about them.

It is very much to be hoped that, after this promising beginning, Woodger will turn his attention to these more troublesome difficulties and will show us what the axiomatic method can do to clarify them. It is possible, however, that logistic can only profitably be employed on a theory which has already attained a certain minimum of completeness, and that the greater part of biological theory has not yet arrived at such a stage. The axiomatic method consists largely in saying as much as is possible without making it clear what one is saying it about. Science, on the other hand, is essentially concerned to make statements about definite objects in the world. The two aims can coincide when the scientist is convinced that the undefined signs of the logistician correspond in fact to entities in which he is interested. The logistician can then provide a language in which science can express precisely those things which it already knows; and this is of course an extremely important function for logistic to fulfil. Perhaps, as many philosophers assert, it is the only function which any type of philosophy fulfils. But scientists know that they are confronted, not only by problems of the technical performance of experiments, but with questions of the formulation of methods of approach, and many of them cherish a belief that philosophy can or should help them in this task. Whether this can be done better by the rigid formalism of logistic than by the older and more tentative types of thinking is a question for the future to decide. Meanwhile all biologists interested in the general aspects of their subject should follow Woodger's pioneering efforts with sympathy and interest. C. H. WADDINGTON.

Random Variables and Probability Distributions. By HARALD CRAMÉR. Pp. 120. 6s. 6d. 1937. Cambridge Tracts, 36. (Cambridge)

The theory of probability has been cultivated in England less for what it is than for what it does. The research of the present century on the theory of estimation and the distribution of statistical coefficients has taken the fundamentals for granted and has sometimes been avowedly non-rigorous. On the Continent, however, there has been strong dissatisfaction with the axiomatic basis of probability, which has manifested itself in different ways. Von Mises has essayed to reformulate the frequency theory of probability by a new theory of admissible infinite sequences of events; Lévy, Fréchet, Kolmogoroff, Cantelli, Tornier and others have replaced the old *a priori* definition of probability as ratio of favourable to possible cases by a more rigorous theory based on sets of points and Lebesgue measure; Hostinsky has developed Poincaré's theory of arbitrary functions.

The tract which we review has come exactly at the time when it was required, as a presentation to English readers of the *a priori* standpoint of Lévy, Fréchet, Kolmogoroff and Cramér himself. The tendency of these writers is towards mathematical abstraction, and the theory that emerges is a branch of pure mathematics, a part of general analysis. An event is described by a set of coordinates, a point or vector, in a space R_k of k dimensions. Possible events form a point-set or aggregate; the probability of an event is a completely additive set function of the associated point-set, and the treatise begins by developing the properties of such functions. The approach thus resembles that of Kolmogoroff's *Grundbegriffe* of 1933 (Springer, Berlin), but is somewhat more specialised in that the event-space considered is a Euclidean space R_k of a finite number k of dimensions.

It will be seen from these brief indications that no time is wasted over discussions of " equal likeliness ". In fact the question does not arise, since to specify the point-set and the Lebesgue measure of subsets is in fact to say what subsets are equally likely. The question of " equal likeliness ", that is,

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of proper choice of the event-space, thus devolves on the *applied* statistician; and the well-known " paradoxes " of geometrical probability show how different choices of event-space lead to different conclusions.

Chapters I and II are devoted to introductory remarks, the definition of Borel sets S in R_k, of completely additive set functions $P(S)$, point functions $g(X)$, and the Lebesgue-Stieltjes integral of $g(X)$ with respect to a set function $P(S)$. The probability function $P(S)$ for a set S is defined as a non-negative completely additive set function such that $P(R_k)=1$. For a set comprising all vectors having components less than those of X, the total probability function $F(X)$, a point function as contrasted with a set function, is called the distribution function of X. Independent random variables are defined as those for which the compound distribution function, in the product space of all the variables, is the product of the several distribution functions.

Chapter II considers distributions in R_1, mean values, moments and absolute moments, and inequalities. Chapters IV and V introduce the characteristic function defined by

$$f(t)=\int_{-\infty}^{\infty} \exp(itX)\, dF(x),$$

with important theorems on the necessary and sufficient conditions for convergence of distribution functions to ensue from uniform convergence of characteristic functions. Chapter V considers the addition of random variables by composition or convolution of distribution functions and multiplication of characteristic functions. This is followed by a theory of convergence in probability, with well-chosen examples on the binomial, Poisson, Pearson Type III and Cauchy distributions. There is also a very useful theorem (Th. 16) on a quotient of random variables.

Chapters VI and VII bring us to the core of the book, the " central limit theorem ", according to which the sum of a large number of independent random variables is distributed approximately according to the normal law. The necessary and sufficient condition of Lindeberg is given, and the sufficient condition of Liapounoff. Chapter VII gives Liapounoff's theorem on the order of the remainder term in the approach to normal distribution and the author's own asymptotic expansion (first given formally by Edgeworth), which differs in the arrangement of terms from the so-called Gram-Charlier, Bruns or Type A Series.

Chapter VIII briefly considers what has been called the *homogeneous random process*, in which the addition of random variables (which we may suppose carried out at discontinuous intervals of time $\Delta\tau$) is replaced by integration with respect to a continuous time parameter τ, and Chapters IX and X extend the important theorems of earlier chapters from R_1 to R_k. The work concludes with a bibliography, confined almost entirely to memoirs later than 1900, and for the most part later than 1925.

As a piece of exposition, the tract must be given the highest praise for its clarity and for the excellent arrangement of its material. It is really the first book to give an adequate account in English of the contemporary tidying up of the purely mathematical side of probability, and should be studied by everyone interested in the postulational basis of the subject. A. C. A.

A Course of Pure Mathematics. By G. H. Hardy. 7th edition. Pp. xii, 498. 12s. 6d. 1938. (Cambridge)

" When Mr. Hardy sets out to prove something, then, unlike the writers of too many widely read text books, he really does prove it. . . . I shall hope to avoid in the future the many weary hours that have usually to be spent

in convincing University students that ' proofs ' which they have laboriously learned at school are little better than nonsense " (Arthur Berry, *Gazette*, V, p. 304, on the first (1908) edition).

" One result, he adds, of all these alterations is to make the book more difficult, and we have the comforting assurance that ' it is no longer necessary to apologise for treating mathematical analysis as a serious subject worthy of study for its own sake ' " (W. J. Greenstreet, *Gazette*, VIII, p. 60, on the second (1914) edition).

" . . . the course is an unsurpassed education in exact thinking, which has played and will long continue to play a notable part in setting a high logical standard for the early training of mathematicians in this country " (E. H. Neville, *Gazette*, XIII, p. 174, on the fourth (1925) edition).

" It is refreshing to compare the grasp of fundamental analytical ideas of the better mathematical students whom I now meet with the slipshod notions and methods of my undergraduate contemporaries and of my earlier pupils " (Arthur Berry, *Gazette*, XIV, p. 428, on the fifth (1928) edition).

Comment on these quotations would be superfluous ; praise of Professor Hardy's book an impertinence. We all know what a change has taken place in the teaching of analysis in England since 1908 and the main cause of that change.

In this new edition the O, o notations have been removed from an appendix and incorporated in the text. Part of the treatment of differentiation has been rewritten ; the author says, " Here I have found de la Vallée Poussin's *Cours d'analyse* the best guide." Examples taken from recent Tripos papers have been added.

The book is probably as fascinating to the pupils of to-day as it was to those of twenty or thirty years ago, but for a different reason ; thanks to the influence of the book itself, there will be less sense of adventure, and more of the firm and masterly filling in of outlines already familiar. But whatever the difference in emphasis, the debt of English students of analysis to the author is beyond computation. T. A. A. B.

Geometrical Optics. An introduction to Hamilton's method. By J. L. SYNGE. Pp. ix, 110. 6s. 6d. 1937. Cambridge Tracts, 37. (Cambridge)

To not a few minds of our day the name of Geometrical Optics suggests a dull and barren realm, devoid alike of interest and of generality, and quite unconnected with any other region of mathematical or physical concern. That this view is passing, though slowly, is due perhaps to the rediscovery of the optical methods and writings of Sir William Rowan Hamilton. And in this connection Professor J. L. Synge, in collaboration with Professor Conway, has performed a useful service by the edition, which jointly they have prepared, of the optical writings of Hamilton, constituting Volume I of Hamilton's *Collected Works*. Here the vast generality of Hamilton's thought emerges, forming as it does the basis not only of optical theory but also of theoretical dynamics, and of the new wave-mechanics.

The fundamental element of Hamilton's method is, of course, the *characteristic-function* which he himself introduced ; " all the problems of mathematical optics " being " considered as reducible to the study of this one characteristic-function ". And in the tract before us Professor Synge sets forth a simple, clearly written and very interesting account of the various forms of this function. The function is introduced, for a series of homogeneous isotropic media bounded by refracting, or reflecting surfaces, as the optical path between two terminal points and as depending upon the coordinates of these points ; and the modifications of the functions are considered as depending upon some, or all,

of the terminal direction cosines. The several functions are calculated for various simple cases—for reflection and for refraction at the surface of a plane, of a sphere, and of a paraboloid of revolution; a valuable addition to the tract. An investigation is given of the arrangement of the rays around a central ray, of the foci and of the focal lines, and of the " aberrations " in the neighbourhood both of the focal lines and of the principal foci. The behaviour of the symmetrical optical system is examined at some length, as an application of the " angle-characteristic ", the function which Bruns, long afterwards, named the *eikonal*; and the first order aberrations are considered—the so-called " five aberrations of von Seidel ", given by Hamilton at a much earlier date. Finally, a chapter is added dealing with rays of light in heterogeneous isotropic media.

The tract should appeal to those interested primarily in Geometrical Optics whether from the theoretical or from the practical side, and it forms an attractive introduction to optical theory. But it should appeal also to any who are interested in the particular application of a vastly general idea, and should be a further aid to a realisation of the general thoughts fundamental to Optics.

G. C. S.

Factor Tables giving the complete decomposition into prime factors of all numbers up to 256,000. By G. Kaván. Pp. xii, 514. 42s. 1937. (Macmillan)

It is rarely that the title of a book can be as completely explanatory as for a factor table, and in 512 pages of this volume, each containing the prime factors of 500 consecutive integers, are given the complete factorisation of all integers up to and including 255,999. On each page the 500 entries are arranged in 10 columns each of 50 lines, consecutive entries being read horizontally. The factors are " centred " in their columns, full points are used to separate factors and primes are printed in heavy type.

Many factor tables covering a far greater range are in existence, but this is the most comprehensive table yet produced to such a high limit; the other tables give either the complete factorisation or only the least prime factor of numbers not divisible by 2, 3 or 5. Fundamentally these skeleton tables provide all the necessary material for factorising large numbers, but in applications, one of which is the approximation by gearing to a given ratio, it is useful to be able to see at a glance all the factors of consecutive numbers. This table, extending the range of such factorisation from 100,000 to 256,000, must therefore be considered a great advance.

Factor tables seem peculiarly liable to errors, which is not altogether surprising when it is considered that copyist's and printer's errors are so difficult to find; but the value of such a table must necessarily depend largely upon its freedom from error. In this table, the description of the method of computation and of the checks applied indicates that considerable care has been taken in its preparation, but a list of 24 errors tends to suggest that not sufficient care was taken in the proof reading. This list of errors and a statement in the *Introduction*, imply that the printed sheets themselves were subjected to a final checking process, although it is understood that the tables were printed from stereotyped plates; it seems strange that either this reading was not done before printing, or if it was, that the corrections were not carried through. Certain samples have been compared with the *Factor Table*, British Association Mathematical Tables, Volume V, and, as the main comparison was with Chernac's *Cribrum Arithmeticum*, all known errors in that volume have been checked; no errors have thus been found.

Although the present edition is published in England, the tables were printed in Czechoslovakia, and it must be stated that the standard of printing

falls far short of that appropriate to the recording of the results of such painstaking work. Opinions may differ as to the use of new style figures (without heads and tails), but it is certainly poor printing that necessitates a list of 38 entries clarifying imperfectly printed figures. The *Introduction* refers to the English edition, suggesting that there is also a Czechoslovakian one, but actually that is not the case and this is so far the only edition.

This volume follows so very shortly after the publication of the British Association Factor Table, which gives exactly the same information for numbers less than 100,000, that comparison is inevitable. That volume was subject to the most extraordinary precautions against error, and the printing was done with great care. For numbers less than 100,000 Kaván's table cannot supersede the B.A. table, but for numbers over 100,000 it should, even with its imperfections, be the standard table for many years to come.

In addition to the English *Preface* (by Dr. B. Šternberg) and *Introduction* (Dr. A. Beer) there is a Latin *Preface* by Professor Petr (dated 1934) and a description of the tables and of their construction in Czechoslovakian by Dr. Kaván. Dr. Kaván started the computations for these tables in 1917 but died in 1933, before they were completed. The completion of a work of this nature, without the guidance or enthusiasm of its originator, must have presented a formidable task, and it is gratifying to note that Dr. Kaván's widow, Mrs. B. Kavánová, took a large share in the work. D. H. S.

The Elements of Mathematical Analysis. I. II. By J. H. MICHELL and M. H. BELZ. Pp. xxiii, 516; xii, 571. 42s. each vol. 1937. (Macmillan)

We have here a course in Calculus written by authors with a mathematical conscience and an ability to teach. Every attempt is made not only to be rigorous and clear but also to explain at some length what is being done. It is a great pleasure to find such a work on calculus. A good illustration of the point of view is in the proof on page 214 of the Lagrange Remainder for the Taylor Polynomial. This is often obtained by a short but very tricky method which confuses the student. Michell and Belz wisely prefer a longer proof requiring more than one application of the mean value theorem. The result obtained is slightly less general but it is quite as useful for the practical determination of error. Moreover, the actual use for numerical work is well illustrated.

The book commences with a very good introductory chapter considering continuity, " one-way ", *i.e.* monotone functions, " dimetric " functions *i.e.* functions of two variables and implicit functions. Upper and lower limits are called " focal bounds " and provide a difficult but probably necessary foundation for further work. There follows a discussion of series, of discontinuities and of orders of magnitude. On page 95 there is a slight deviation from normal practice. It is stated that "$f(x)=O(x^n)$ is sometimes used to express that $f(x)$ is of the order of x^n " and by this the authors mean that $f(x)/x^n$ has its upper and lower limits of the same sign.

Then come chapters on differentiation, successive derivatives and Rolle's theorem and applications to curvature, trigonometric, logarithmic and exponential functions. It is doubtful whether any advantage is gained by laying so much stress on the epicene and expocyclic functions.

A final chapter on special curves and polar coordinates completes the first volume.

The second volume is concerned with integration and elementary differential equations. The reviewer would like to see generally accepted the use of " terminals " in place of " limits of integration " which are not limits in the ordinary sense. On pages 532 to 535 the proof of the existence of the sum-limit for a continuous integrand is too condensed and would need a little expansion

by the teacher to form a completely rigorous whole. This is not a serious matter however.

In addition to what is traditionally found in a course on integration there is a chapter on the approximate representation of functions by the method of moments, and orthogonal functions and another chapter on the integral form of the Taylor polynomial remainder. The latter is too often omitted in English books. There is a final chapter or appendix on the arithmetical basis of the calculus.

There is one obvious difficulty in recommending these two volumes, namely their cost and bulk. On the other hand, certain other bulky and expensive books on the same subject are known which are completely rubbish and which are still used in schools in spite of their being many years out of date. These two volumes give numerous well-chosen examples and ample illustrations. They serve all the purposes served by the other books and do not make heavy reading while at the same time they do not kill the student's innate sense of logic.

These two books by Michell and Belz are strongly recommended not only to university students but still more perhaps to mathematical boys (and girls) in school. At least one copy should be in every school library.

I also venture to ask all examiners for scholarships to set a certain number of questions each year of a type such as could only be answered by those who had studied the calculus either from books such as these or from teachers' notes of a similar character. Perhaps there will come a time when bright scholars will arrive at the university with the knowledge that manipulative skill is not enough. P. J. D.

Modern Theories of Integration. By H. KESTELMAN. Pp. vi, 252. 17s. 6d. 1937. (Oxford)

There is not, as far as the reviewer knows, any book in English which fills the gap filled by this book, namely one giving a straightforward account of the important parts of Lebesgue Integral theory without frills. Hobson is interested in many side-lines not wanted by the ordinary mathematician. Many other books branch off into the most recent generalisations and applications, involving a more advanced point of view.

After a preliminary discussion of sets of points restricted to what is necessary for the purpose of the book, there follows an account of the Riemann integral. This is well chosen in not going too far along dead branches, but the account is thorough on the fundamental aspects.

The next three chapters tackle Lebesgue measure and measurable functions, the author having picked out simplifying paths from various sources and invented some himself. The integral in R_n is found as a measure of a set in R_{n+1}.

At the beginning of Chapter V I have a small private quarrel with the author. Kestelman states that " the marked simplicity of the new integral is a reflection of the broad and simple properties which characterise the Lebesgue theory of measure ". I prefer to follow W. H. Young in regarding the Lebesgue integral as defined specifically to ensure that for bounded sequences

$$\lim \int f_n dx = \int (\lim f_n) dx.$$

The measure of a set is then simply the integral of the characteristic function of the set. However, it must be admitted that general practice and public opinion are against the reviewer. It is always difficult in mathematics to decide which is the cart and which the horse. It is possible, for example, to imagine a geometry in which the parallel postulate is a theorem implied by Pythagoras' theorem taken as a postulate.

Chapter VI not only considers functions of any sign but also complex functions. Incidentally, the adjective "summable" is used instead of "integrable L".

Chapter IX is about extensions of one-dimensional integrals such as the Denjoy integral, and Chapter X applies the previous theory to Fourier Series.

The chief impression left after reading the book is the remarkably good power of choice shown by the author. Not only are proofs clear and to the point, but no attempt has been made to show off or to go unnecessarily into advanced branches of special interest to the author.

It is a sound piece of work altogether and strongly to be recommended to those who have not as yet been educated as far as the Lebesgue integral.

P. J. D.

Newton's Principia in 1687 and 1937. By T. M. Cherry. Pp. 28. 1s. 1937. (Melbourne University Press; in association with the Oxford University Press)

Although the easier portions of Newton's *Principia* were prescribed for study in some universities until comparatively recent times, I feel sure that many of the younger mathematicians have very vague ideas about the scope of Newton's work and the methods by which he achieved his purpose; I certainly had until this small book by Professor T. M. Cherry, of the University of Melbourne, came into my hands.

The book contains a slightly amplified version of a lecture given by Professor Cherry to commemorate the 250th anniversary of the publication of the first edition of the *Principia*. In it, he gives a compact yet clear and readable account of the dynamical problems which Newton solved and of the mathematical technique he used, and concludes with a brief discussion of the changes produced in Newtonian dynamics by Einstein's Theory of Relativity. It must have been a most stimulating lecture.

E. T. Copson.

An Introduction to Mathematics. By J. C. Hill and W. C. McHarrie. Pp. 128. 2s. 1937. (Oxford)

Here is an outstanding book. It is one of a group of four "designed as a balanced introduction to Geography, History, Science and Mathematics. The underlying idea of the series is twofold: first to gather up the interests of the pupil, secondly to help him to develop those interests along practical lines." This is an admirable programme, and it is well carried out: though intended for 8-year olds the book forms a genuine introduction to *mathematics*. "The children recapitulate the history of the subject and move swiftly to mathematical work of importance." (The quotations are from the prospectus.)

The book consists entirely of forty-nine exercises, each containing twelve questions, and the work is carefully graded. The varied topics are all integral with the scheme, and include astronomy, early systems of numeration, derivation of terms used, position finding, areas of circle and sphere, angles of a triangle, angle in a semicircle, "Pythagoras", graphs, simple series. Considerable use is made of systematic and subtle repetition, allowing the subconscious mind to assist in developing each theme. As the authors say "a psychological rather than a logical sequence is followed, and what may be regarded as an unconventional arrangement is the result of a careful experiment". The procedure recommended (in a foreword to the teacher) is worth quoting almost in full:

"1. Let each pupil work at his own pace. There are plenty of other books which can be given to children when they finish this one.

2. Don't explain anything unless a child asks for an explanation, and don't

stop the whole class because a few ask for help. Give individual help in a whisper so that others are not disturbed, or call a group round the blackboard. Rest the children's ears during the lesson so that you get good attention for your next oral teaching.

3. Go round the class quietly and make sure that all pupils have good tools to work with. Many school compasses require tightening; many compass pencils are too long.

4. See that squared paper is available ".

The exercises are very easy, and it would be necessary to insist on careful and complete answers, as a quick pupil could rush through them without properly digesting the course. No misprints have been noticed, but in Ex. 48, No. 12, is shown an intentionally equilateral triangle with side 8″ and height 7″, but no hint of the approximation in spite of the fact that a proof of " Pythagoras " occurs earlier in the same exercise.

The book is not complete in itself and is clearly supplementary to a normal course in arithmetic. There is much attractive geometry, but little algebra other than simple formulae and generalised arithmetic; definitely not " the mixture as before ".

The publishers are to be congratulated on a beautiful production. The numerous diagrams are large and clear, the type bold and the cover attractive. Every teacher of junior mathematics should get hold of a copy, for though designed for the very young its language is not too childish for pupils of twelve or over: it might be regarded as food for infants and a tonic for (mathematically) spiritless children. None brought up on this book could fail to regard mathematics as an essentially human study growing out of social needs. A. P. R.

Revision Exercises in Elementary Mathematics. By JOHN STEPHENSON. Pp. 112, 15. 1s.; with answers 1s. 2d. 1937. (Oxford)

This book is intended for Senior and Central Schools, but would be equally suitable for other pupils of 11 +. It contains three dozen sets of papers, each set consisting of C, B, A papers in gently ascending order of difficulty. Each paper has a separate page, and though the type is large and clear there is ample room for the eight questions. The diagrams, however, are so small that any one can be completely covered by a postage stamp, and the letters or figures on them are difficult to read.

The C and B papers include very easy fractions and decimals, simple equations, formulae and " arithmetic with letters ". The A's in addition contain square roots, percentages and simple interest, areas and volumes, " Pythagoras ", simple simultaneous equations, and numerical questions on angles and parallels. The later A papers have an occasional question on logarithms or trigonometry.

There is an air of unreality about the book, surprising in one designed for modern schools with a practical bias. There are no formulae other than those used in the mensuration questions, the geometry is abstract, with no mention of position-finding, scale drawing or even pattern drawing. The trigonometry consists of such questions as " If $\sin A = \frac{4}{5}$, find A ", and the answers are given to the nearest degree. There is not even a tower or a cliff whose height is to be found. In arithmetic there is no suggestion that statements like " 1 litre = $1\frac{3}{4}$ pints " are approximate; and all decimal divisions terminate. In B. 14.3 occurs the phrase " speed per hour "—probably a misprint, and in A. 27.3 the right-angles should be marked.

The book would provide useful revision for Common Entrance candidates in Preparatory Schools, but it is not a very inspiring compilation. A. P. R.

Elementary Theory of Operational Mathematics. By E. Stephens. pp. xi, 313. 21s. 1937. Electrical Engineering Texts. (McGraw-Hill)

This volume deals mainly with the type of operator treated by Boole in his book on Differential Equations (D.E.), and some extensions thereof. The fractional operators used by Heaviside, and operational processes based on the Laplace transform and the Mellin inversion theorem are not treated. Starting with $D \equiv d/dt$, various theorems are stated (not proved) and applied to the solution of ordinary *D.E.* with *C.C.* Using partial fraction theory, a proof, by analogy, of Heaviside's expansion theorem is given. It is not applied to any type of problem. This is followed by chapters on, (*a*) determinants, Wronskians and Jacobians, and (*b*), matrices. The formulae are useful for reference purposes, provided the reader is acquainted with the underlying theory and the more elementary parts of these subjects. Applications are made to the solution of systems of ordinary linear *D.E.* with *C.C.* Operational processes are then extended to three variables, many useful theorems and formulae being given. In Chap. XI, series solutions of Bessel, Legendre, Hermite, and Hypergeometric equations are found operationally. The general linear *D.E.* and systems of linear *D.E.* are also treated. Chap XII is devoted to the construction of *D.E.* in mathematical physics (and dimensions pertaining thereto), whilst boundary conditions are dealt with in Chap. XIII. There are numerous examples in each chapter to be worked out by the reader, but unfortunately the answers are not given. The main formulae in the text are collected in Appendices I, II, whilst Appendix III contains historical notes and theorems due to Boole, Graves, and Murphy.

From a mathematical viewpoint—apart from absence of proofs which are given in other works to which reference is made—the book is a highly commendable work. The subject matter has been carefully selected and is well presented. Neither the text nor the examples are couched in technical language, so that mathematicians teaching technical students will find it extremely useful.

The author says, however, that the book has not been addressed to mathematicians. " It is written primarily for all those who live and work to make mathematics useful to mankind, because of the inherent simplicity and beauty of the operational forms, and because of their wide application to the daily tasks of the teacher and worker in our engineering schools and industrial laboratories ". It is a pity that the author included the latter part of this sentence in the preface, because the type of text and the prior knowledge needed to read it intelligently is beyond the engineer or industrialist. For him the text should have contained more worked examples relating to engineering, and a practical atmosphere should have been introduced. Some of the terminology is unfortunate, *e.g.* " Operations on zero ". In his practical way the engineer will think, how can one operate on what does not exist? The terminology is borrowed from others who (unfortunately) addressed themselves to engineers. It neither says what it means, nor means what it says! The engineer will need to study the elements of determinants, Jacobians, and matrices before he can follow the text on these topics. On pp. 194, 201, the author introduces the terminology of the complex variable and speaks of " poles ". It is asking too much of the engineer to be acquainted with " singularities ", and a brief explanation should have been given. On the whole it is felt that the engineer, technician, or industrialist who works his way through the book will need assistance from the mathematician.

In Appendix 3, the historical summary includes not only the Boole, Graves, Murphy operational procedure, but that of Heaviside and his successors. Of Heaviside the author writes : " It appears, however, that he was either not

aware of the complete body of theory on this subject, which had been in existence more than twenty years before he wrote, or he drew freely, though inadequately, upon it and gave no one credit ". This sentence should not have been printed. A little research and the realisation of Heaviside's *main* contribution to the subject of operators would have prevented such a thought. In his first R. S. paper (1893), Heaviside quotes from Thomson and Tait's *Natural Philosophy* p. 197 : " The investigation of this generalised differentiation presents difficulties which are confined to the evaluation of P_s [spherical harmonics], and which have formed the subject of highly interesting mathematical investigations by Liouville, Gregory, Kelland, and others . . . ." Then Heaviside goes on to remark ". . . . not having access to the authorities quoted, I was compelled to work it out myself. I cannot say that my results are quite the same, though there must, I think, be a general likeness ". This answers the author's statement. It must be realised that for an impecunious person like Heaviside who worked independently and was unattached to an educational institution, reference facilities—particularly half a century ago—were practically nil.

Liouville solved equations of the type :

$$\frac{d^{\frac{1}{2}}y}{d(\sqrt{x})^{\frac{1}{2}}}+\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=F(x), \quad \ldots\ldots\ldots\ldots 1$$

$$\frac{dy}{dx}-\frac{2d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+y=0, \quad \ldots\ldots\ldots\ldots 2$$

but no symbolic operator was used, and the initial or boundary conditions were not absorbed into the solution. These ideas were due to Heaviside. The concept of solving physical problems by writing p for d/dt and algebrising by aid of the formula $p^{-\nu} \doteqdot t/\Gamma(1+\nu)$, was originated by Heaviside, who assumed such procedure to be legitimate, because it always worked. The legitimacy was proved by Bromwich in his memorable paper of 1916, and also by Wagner in 1916. Neither of these authors is mentioned in Appendix 3, and we hope that the omission is unintentional. At the end of the historical notes the author says : " Below is given a *complete** bibliographical list of *all* papers and works on operational methods *and applications* from their beginnings in 1765 to date ". This is incorrect, for works published prior to 1937 by the following are missing† : Bromwich (7), Carson (3), K. Dhar (3), S. C. Dhar (1), Doetsch (4), Goldstein (2), Hartree (1), Humbert (7), McKay (4), McLachlan (3), van der Pol (2), Varma (2), Wagner (2), and thirty other authors (43)—a total of 84 papers, but there may be more. The addition of these would bring the total number of references up to about 430. Two misprints have been noted : In the third reference from the foot of p. 290, 1826 should be 1926, whilst Forsyth has no *e* at the end.

N. W. McLachlan.

Introduction to Mathematical Probability. By J. V. Uspensky. Pp. ix, 411. 30s. 1937. (McGraw-Hill)

This book appears to be the most complete work on the subject yet produced in the English language. Much important recent work in probability is due to Russian authors, whose work is little known in Western Europe owing to language difficulties. Professor Uspensky, a professor at Stanford University, California, presumably has Russian connections ; at any rate, he makes great use of Russian sources, including the older writers Tshebysheff, Markoff, Liapounoff, Tschuprow, and those still active, such as Bernstein, Khintchine,

* The reviewer's italics.

† The figures in brackets indicate the number of papers missing.

Romanovsky, and Kolmogoroff. Another distinctive feature of the book is the proof of the theorems, due to Karl Pearson, " Student " (W. S. Gosset), and R. A. Fisher, which are of fundamental importance in statistical tests of significance.

The preface tells us that the book is arranged so that the first twelve chapters, with the possible exception of Chapters VII and VIII, are accessible to a person without advanced mathematical knowledge. It would be more correct to say that large portions of these chapters require only elementary algebra, but that other portions may involve integration, continued fractions, hypergeometric series, Hermite's polynomials, complex variables, and other comparatively advanced mathematics. The last four chapters incorporate the results of recent researches, and require more mathematical knowledge and a mature outlook.

The introduction, which is not up to the high standard of the rest of the book, gives a brief account of various views on the nature of probability and the history of the subject. The author has a low opinion of modern attempts to build up the theory as an axiomatic science. They " may be interesting in themselves as mental exercises ; but from the standpoint of applications the purely axiomatic science of probability would have no more value than, for example, would the axiomatic theory of elasticity ". The old *equally likely* definition is preferred, with the hope that some day, as in the case of Mendelism, it will be discovered how to apply it beyond its apparently narrow limits. " After all, is not faith at the bottom of all scientific knowledge? "

Chapters I, II, and III cover the usual elementary course in probability, with important supplements. In Chapter I we find a characteristic point of view, which occurs again and again in later chapters. It is not sufficient to find an exact algebraic solution of a problem, because that solution is usually unfitted for numerical computation with large numbers. Two approximate formulae are found, such that both are easy to evaluate numerically, and the exact solution lies between these two values, which are very close together. Chapter III contains Markoff's application of continued fractions to such inequalities. Chapter IV deals with Bayes' Theorem and Laplace's Law of Succession. Here, as elsewhere in the book, it is clearly pointed out what conditions are necessary for the validity of the results. Chapter V explains the use of difference equations in probability problems, such as those of runs of successes. Partial difference equations are solved by Laplace's generating functions, and also by an elegant but less known method due to Lagrange. Chapter VI deals with Bernoulli's Theorem, giving a modernised version of the original proof and a warning against the illegitimate applications of the theorem. Chapter VII deals with the approximate evaluation of Bernoulli probabilities for large numbers, a difficult problem attacked by De Moivre and Laplace. It is pointed out that it is easy to obtain Laplace's approximation if, as is usual, the magnitude of the error is not examined. Professor Uspensky will not tolerate such laxity, and gives a long and detailed investigation (over nine pages) culminating in an inequality. Chapter VIII deals with the ruin of gamblers which may occur even in favourable circumstances owing to shortage of capital. Chapter IX (Mathematical Expectation) is preliminary to Chapter X (The Law of Large Numbers). Chapter XI applies this law with experimental tests of the results. It also deals with the Divergence Coefficient, which is related to Tschuprow's quotient Q. Chapter XII passes from the discontinuous to the continuous, and introduces the important *characteristic function* of distribution, which is of fundamental importance in advanced statistical work. This leads on naturally to the more advanced part of the book. Chapter XIII explains Stieltjes' integrals, which include ordinary

integrals and finite sums, thus dealing with both continuous and discontinuous distributions in a unified way. It also contains Liapounoff's inequality for moments, and his ingenious proof that the distribution is uniquely determined by the characteristic function. Chapter XIV deals with fundamental limit theorems, such as Laplace's derivation of the normal law. Laplace's own attempt to prove this important theorem was not rigorous, and could not easily be made so. The same is true of the attempts of Poisson, Cauchy and many others. Only after many years were sound proofs obtained, first by the Russians Tshebysheff (1887), Markoff (1898) and Liapounoff (1900-1). Later proofs by Lindeberg and Levy make use of a device due to Liapounoff, a fact often overlooked by non-Russian writers. Professor Uspensky gives Liapounoff's method in this chapter, and Tshebysheff's and Markoff's methods in Appendix II. Chapter XV deals with the normal distribution in two dimensions, including an account of the theory of Pearson's χ^2 test, with the warning that the lack of information as to the error incurred by using an approximate expression renders the application of the test somewhat dubious. Chapter XVI finds the distribution of various functions arising from a number of normally distributed variables, such as their mean, variance, and "Student's" quotient. We are also given Fisher's investigation of the distribution of the correlation coefficient, and his transformation into another distribution which is nearly normal, with the author's usual note of caution about the limit to the legitimacy of applications. There are three appendices, of which the second has already been mentioned. The first deals with Euler's summation formula, Stirling's theorem, and some definite integrals. The third gives a letter of Gauss, dated 1812, in which he mentioned a problem he could not finish. He expressed his certainty that Laplace would find a more complete solution in a few moments. As a matter of fact, the first satisfactory solution was found in 1928 by R. O. Kuzmin, whose work is given in this appendix. The book concludes with a table of the probability integral and an index.

If it were not for the almost simultaneous publication of the Cambridge Tract, *Random Variables and Probability Distributions*, by H. Cramér, Professor Uspensky's book might have been described as unique, as far as the English language is concerned. However, Cramér's tract, though dealing with some of the same problems as the later chapters of Uspensky's book, does so in a more sophisticated manner suitable to those who already have considerable knowledge of the subject. Thus from the point of view of young English students who are beginning to specialise in the theory of probability Uspensky's book can be recommended as indispensable. A valuable feature is that at the end of every chapter is a collection of illustrative problems, some easy examples of the theorems in the text, but others important theorems supplementary to the text. In all such cases full outlines of solutions are given.

H. T. H. Piaggio.

Intermediate Algebra. By S. E. Urner and W. B. Orange. Pp. xv, 432. 12s. 1937. (McGraw-Hill)

Like *College Algebra* by Palmer and Miser reviewed in the *Gazette* of October 1937, this book does not seem to fit any English curriculum.

In 386 pages it gives a course of algebra beginning with a discussion of the fundamental laws and the simple operations in arithmetic and algebra and including an introduction to the binomial theorem and determinants. The trigonometry of the right-angled triangle and an introduction to annuities with the actuarial notation also appear.

In the introduction the authors express the feeling that the idea of the function of the relations of variable quantities should dominate the whole

study—but confess themselves impeded (*sic*) because of the certainty that prospective users of the book might be afraid of anything unfamiliar. After this confession the reader is prepared for such passages as :

" First, what is meant by taking a number (-1) times? We have seen that -1 is just like $+1$, except for direction. So, taking a number 1 time is just taking it as it is ; taking it (-1) times would quite naturally be taking it in reverse direction ; that is, $1 . 4 = 4$ and $(-1) . 4 = -4$." F. C. B.

Integralgleichungen. Einführung in Lehre und Gebrauch. By G. Hamel. Pp. viii, 166. RM. 9.60 ; geb. RM. 12. 1937. (Springer, Berlin)

This book is founded on a course of lectures given recently in Berlin. The audience consisted mainly of physicists and engineers, and Professor Hamel's object was to teach them as much analysis as they needed in order to apply the theory of integral equations. One wonders how many English physicists and engineers would stand this sort of thing, but at any rate all mathematicians learning the theory would do well to read Professor Hamel's book. It aims at teaching the theory primarily by means of examples. A series of these, taken from potential-theory, electricity, vibrating membranes and so on, shows clearly what the problems are, and what sort of solutions are to be expected in simple cases. When a general theorem is suggested it is stated, and the proof deferred until the general theory is complete. There follows a good account of the classical theories of Fredholm and Schmidt, with a brief sketch of the method of Hilbert and its relation to linear equations with an infinity of unknowns. The book concludes with a number of further special problems and singular equations, Mathieu's equation, Abel's equation, and problems occurring in aerodynamics.

There must be many readers who will find all they want of the general theory in this book, and all will find the examples stimulating and instructive.

E. C. Titchmarsh.

Differential Calculus. By S. Mitra and G. K. Dutt. Pp. xiv, 302. 10s. 1937. (Heffer, Cambridge)

This book consists of three parts. The first deals with such matters as variables, functions, and limits ; the second, with differential calculus proper ; the third, with geometrical applications.

The first part is too condensed to serve any useful purpose except to refer the student, as it frequently does, to Professor Hardy's *Pure Mathematics*. Indeed much of this part is little more than a watered version of parts of that book. For example we find the headings : Idea of a Function, Graphical Representation, Polar Coordinates, A. Polynomials, B. Rational Functions, C. Explicit Algebraic Functions, D. Implicit Functions, Transcendental Functions, Other Classes of Transcendental Functions, together with some of the corresponding text, exactly as in Professor Hardy's book.

The wise student will prefer to read the original, with all its wealth of illustration.

It should be added that certain topics are included which have become prominent in lectures and tripos papers at Cambridge since the time when *Pure Mathematics* first appeared. The sources of these sections are sufficiently indicated in the Preface.

The Differential Calculus section, although it would not be suitable for a schoolboy beginner, might meet the needs of one who had started the subject late in life and had acquired some knowledge of the subjects mentioned in the first part of this book. The geometrical portion calls for little criticism, but the treatment of elementary differential geometry is too concise. In two

or three places the Integral Calculus is required and reference is made to a forthcoming volume. It is probably not satisfactory to deal with the two subjects in successive volumes in this way. A. R.

Mathematical Snack Bar. A Collection of Notes and Results. By N. Alliston. Pp. viii, 155. 7s. 6d. 1936. (Heffer, Cambridge)

No one can be interested in mathematics for a lifetime without making inventions and discoveries of his own, though they may be nothing more than neat solutions of questions to which no clue was attached, unconventional tricks of exposition, illuminating examples, or gleanings in a field from which the reapers have passed on. Naturally we want to see our work in print; permanence is one of the characteristics of mathematics, and it is proper for us to feel that what we have found out need not be found out again. Mr. Alliston, to whose notes any of the descriptions just given would do less than justice, was a steady contributor to the mathematical columns of the *Educational Times* and to the volumes which continued for a while the series of reprints from those columns, but instead of finding in the *Gazette* or elsewhere a satisfactory substitute when those half-yearly " volumes in continuation " ceased, he has accumulated his material to offer it with a disarming modesty as an independent book.

" The subject-matter ranges over the Euclidean and the Diophantian, and touches on the theory of numbers. The treatment may be said to be elementary, for although our course has led us upon occasion into rather deep waters, higher mathematics have not been invoked." Only an annotated expansion of the table of contents could describe the variety of the fare. The dominant flavour is Diophantine: not only is much of the geometry concerned with triangles that are rational in various respects, but when the geometer constructs a triangle from the perimeter, the height, and one side, or cuts off equal corners from a given triangle by a right-angled wedge pivoted on the base at a given point, he is not temperamentally far removed from the arithmetician who finds values of a variable to make the values of two given functions to be simultaneously perfect squares or primes. But any generalisation would mislead; the menu includes a geometrographical comparison of various constructions for the circles which touch three given circles, comments on Goldbach's guess, and a dozen other dishes from which the Diophantine flavour is absent.

Mr. Alliston takes full advantage of his freedom to hazard a conjecture without going beyond the *prima facie* evidence, or to pick up a topic at any convenient point and to drop it without ceremony. In one case at least the lack of curiosity or of enterprise is astonishing: having found an approximate formula for the number of prime couples separated by 2 below an assigned limit, he compares the values given by his formula for a few small ranges with counts made by Glaisher, and leaves his estimates of the numbers in the first and second millions with the remark " No count is available to check these totals "; from Lehmer's table of primes he could have made the count himself in a few hours. The sequel to the account of this problem is illuminating. " Essentially," says the author, " to find pairs of primes separated by 2 is to find values of n such that $2n-1$ and $2n+1$ are both prime. Why not look for values of n such that $6n+5$ and $10n+3$ are both prime? " In the absence of a guiding principle the only response is " Why not? " and the theory of numbers is seen to be the subject which reveals most starkly that mathematics is the study of questions that interest mathematicians. *

* Compare G. B. Jeffery in his Presidential Address to the London Mathematical Society: " ' Pure mathematics is a good subject because I like it ' . . . might appear to be a better answer than any of the others ". (*Journal*, vol. xiii, p. 74)

The printing and display are excellent, but the use of a comma as a decimal point and of a point raised above the line as a sign of multiplication has no obvious advantages. Misprints and slips are rare, and the writing has a clarity which if it owes something to the irresponsible nature of the book is none the less pleasant. We can hardly urge our readers to follow Mr. Alliston's example, for we do not want to wait a quarter of a century for the opportunity to profit by work which has been done ; but to readers who share his tastes his catering may certainly be recommended. E. H. N.

Anekdoten aus dem Leben Deutscher Mathematiker. By J. MAHRENHOLZ. Pp. iv, 44. RM. 1.20. 1936. Mathematisch-Physikalische Bibliothek, Reihe I, 18. (Teubner)

The teacher who wishes to stimulate his mathematical pupils by the use of incidents from the history of mathematics, a device whose employment was admirably described by Mr. Siddons in the *Gazette*, May 1937, must always be collecting suitable material. He will be able to glean a good deal from this little book, which deals with Adam Riese, Stifel, Kepler, Weigel, Leibniz, Euler and Lambert, Kästner, Gauss, Abel's visit to Berlin, Steiner, Weierstrass, Schellbach. T. A. A. B.

Grundlagen der Geometrie. By G. THOMSEN. Hamburger Mathematische Einzelschriften 15. Pp. viii, 88. Rm. 4.50 (wraps), 5.50 (cloth). 1933. (Teubner)

The ideas which are developed formally and systematically in this booklet are those of which Professor Thomsen himself wrote an account for the *Gazette* (vol. XVII, p. 230) shortly before his lamented and premature death ; it is therefore unnecessary to explain them in this notice. Professor Thomsen's is the latest attempt to realise the schoolboy's dream of a geometrical machine ; unlike earlier attempts, it is based directly on the groups which characterise motion in the Euclidean plane and in Euclidean space, but as in so many intrinsic systems the operations of the symbolic calculus are unattractive and hard to remember.

We may pertinently ask whether there is any sense in which the schoolboy's dream could be realised. Suppose, for example, that the orthocentre H of a triangle ABC is involved. The relation between the four points A, B, C, H is symmetrical, but in elementary geometry it figures always as a symmetrical synthesis of the three unsymmetrical relations $AH \perp BC$, $BH \perp CA$, $CH \perp AB$, and whatever the formal expression of the relation in a calculus may be, we must be able to read from it the expressions of these three relations. Of these relations, two imply the third ; in a particular problem, two are connected with the hypothesis and one with the conclusion, and if the calculus cannot indicate which two come first, the difficulty of constructing a proof is precisely the same as with classical methods. Mention of the orthocentre suggests further consolation for the schoolboy. The theorem asserting the existence of the orthocentre is, that if $AH \perp BC$ and $BH \perp CA$, then $CH \perp AB$. Now it is patently impossible for a machine to turn out results about lines which have not been fed into it, and the lines CH, AB in the conclusion seem never to have been inserted. The fact is that we have omitted from our expressed hypotheses the four concurrences, of AB, AC, AH, of BC, BA, BH, of CA, CB, CH, and of HA, HB, HC, which are implicit in our notation and essential to our result. Speaking generally, if we had a machine, we should have to supply it with all the data, implicit as well as explicit, or at least with all the relevant data. If we have still to acquire the knack of picking out the relevant data, we are not much better off with a machine

than without; on the other hand, in a complicated figure the mass of implicit data is suffocatingly large.

Professor Thomsen was a stern critic of his own calculus, and he was by no means satisfied with the form to which he had brought it. The work was intensely individual, and it is unlikely that anyone else will develop it exactly as he would have done; that is all the more reason why every geometer who is interested in the groundwork of the subject should read in the author's own words the story of a bold intellectual experiment. E. H. N.

The Application of Moving Axes Methods to the Geometry of Curves and Surfaces. By G. S. MAHAJANI. Pp. viii, 60. 1937. (Aryabhushan Press, Poona)

In publishing an essay submitted for an undergraduate prize in 1923, Dr. Mahajani seems to have overlooked the possibility that if the essay was not awarded the prize perhaps it was not a good essay. The first two chapters, on the fundamental formulae of moving axes and their application to curves in space, are not altogether unsatisfactory, in spite of a pretentious and unintelligible discussion of " translational drag "; if the booklet had stopped there, we might have wondered if it was superfluous, but we should have agreed that it might serve a good purpose in making students inquisitive as to the power of kinematical methods. But the third chapter, which deals with curves on surfaces, is too scrappy and too ill-designed to arouse anyone's interest; for example, geodesics are used before they are defined, and the deduction that geodesic curvature is unaltered by deformation is drawn from an assertion regarding spin-components for which no shadow of reason is given. It is true that there is room for a stimulating introduction to Darboux, but we are sorry that Dr. Mahajani has risked in this field the reputation which he has earned as a teacher of analysis. E. H. N.

Grundgedanken der Teiltheorie. By E. FORADORI. Pp. 79. RM. 4.80. 1937. (Hirzel, Leipzig)

The theory of sets has to do with sub-sets of a given aggregate of individuals. The fundamental relation is $x \epsilon X$, to be read " the element x belongs to the sub-set X ". If $x \epsilon X$ implies $x \epsilon Y$, where X and Y are given sub-sets and x is a variable individual, the set X is contained in Y: in symbols, $X \subset Y$. Clearly $X \subset X$, and $X \subset Z$ if $X \subset Y$ and $Y \subset Z$. These last two conditions, abstracted from the theory of sets, form the basis of the book under review. One starts with a domain consisting of undefined " terms " and an abstract relation $a\mu b$, to be read " the term a is part of, or is contained in, the term b ", between some or all of the terms. The only initial conditions imposed on the relation μ are that it shall be reflexive and transitive. That is to say, $a\mu a$, and $a\mu c$ if $a\mu b$ and $b\mu c$. If $a\mu b$ and $b\mu a$ one writes $a \equiv b$. Clearly the relation $\equiv$ is reflexive, symmetric and transitive. Thus the terms fall into mutually exclusive classes, two terms a and b belonging to the same class if, and only if, $a \equiv b$. The terms in each of these classes are now to be regarded as formally identical. With this understood, one can, under certain conditions, define the " union " and the " intersection " of two terms a and b. The union, if it exists, is a term $a+b$, characterised by the properties $(a+b)\mu q$ if $a\mu q$ and $b\mu q$, and if $p\mu q$, where $a\mu q$ and $b\mu q$, then $p\mu(a+b)$. The intersection, if it exists, is a term ab, characterised by the properties $q\mu ab$ if $q\mu a$ and $q\mu b$, and if $q\mu p$, where $q\mu a$ and $q\mu b$, then $ab\mu p$. Thus union and intersection are dual terms, each being obtained from the other by interchanging " is contained in " and " contains " throughout the definitions. The union and intersection of the terms in any domain are similarly defined and, if they

exist, they are unique (remembering that $\equiv$ is to be interpreted as the sign of formal identity).

These ideas form the basis of a definition and broad classification of continua which occupy the greater part of the book. The idea of continuity appears first in the definition of a "continuous nesting" (*kontinuierliche Schachtelung*). A "nesting" is a domain S, such that either $a\mu b$ or $b\mu a$, where a and b are any terms in S, and it is "continuous" if:

(i) each term in S is the intersection of all those terms of which it is a proper part (a is a proper part of b if $a\mu b$, $a \not\equiv b$);

(ii) to each nesting T, whose terms are contained in S, corresponds a term of S which is the union of all the terms in T.

Then a continuum is defined as a domain k, such that each nesting in k is contained in a continuous nesting in k. This definition is given in Chap. II, together with some generalities concerning continua. In Chap. III they are divided into classes, a continuum of the nth class being n-dimensional, and studied in greater detail by methods somewhat similar to those used in combinatorial analysis situs. The final chapter contains a brief account of the continua which are used in analysis.

In the first few pages of Chap. I there is, it seems, an attempt to read too much into the initial conditions imposed on the relation μ. For example, on p. 8 one finds a fairly sophisticated "Gegenbeispiel", showing that a need not coincide with b if $a \equiv b$. This is surely rather ponderous, considering that μ may be any reflexive transitive relation (*e.g.* "compatriot of" between human beings) and $a \equiv b$ simply means that μ is symmetric between a and b. Moreover, it should have been stated explicitly on p. 8 that a is to be formally identified with b if $a \equiv b$. As it is, one is left to infer that $\equiv$ means formal identity from the proof that the union $a+b$ is unique, if it exists (p. 9), a deduction which is not confirmed until p. 18. But, apart from this initial obscurity, the book is well written and it is remarkable how many interesting theorems follow from the very simple assumptions. J. H. C. W.

Cussons' Mensuration Charts. Compiled by G. H. Grattan-Guinness. (Cussons, Manchester, 1937)

These charts, four in number, give in clear illustration, carefully labelled and in bold statement, the appropriate formula with its mode of derivation for the areas and volumes of the various shapes and solids that the boy or girl will meet in his ordinary school course.

Chart I. Rectilineal areas, the rectangle, the parallelogram, the triangle, the trapezium and the quadrilateral. There is also a simple explanation of π.

Chart II. Circular areas—the circle, the ring, the surfaces of the cylinder, the cone and the sphere.

Chart III. The fundamental volumes—the rectangular box, hence the cube and the right prisms (the square, triangular, circular with the hollow pipe, the polygonal and the general), and from these to the corresponding pyramids, with $V = Ah$ as the general formula for the prisms.

Chart IV. (a) Circular volumes—the cylinder and the pipe, the cone, the sphere and the relationship $3:2:1$ of the volumes of the cylinder and corresponding sphere and cone. (b) The oblique solids—the prism and the pyramid.

These charts are 40 in. × 30 in., a very convenient size for wall or blackboard use, and are on paper at 7s. 6d. each (£1 5s. for the set of 4); on linen, 10s. 6d. each (£1 15s. for the set); and on linen, varnished and with rollers, at 12s. 6d. each (£2 5s. the set).

While it may be true to say that the boy or girl can gather all this information from a text-book, these charts provide all the information collected and clearly set out so as to impress itself upon the mind of the observer. They are almost self-explanatory, and they help the pupil materially in his revision. As charts are helpful to the chemist, so this set may be regarded by the mathematical teacher as a very useful addition to the equipment of his mathematical laboratory. E. J. A.

Sur les théorèmes inverses des procédés de sommabilité. By J. Karamata. Pp. 47. 12 fr. 1937. Actualités scientifiques et industrielles, 450 ; exposés sur la théorie des fonctions, VI. (Hermann, Paris)

Abel's theorem on power series is that if the series $f(x)=\Sigma a_n x^n$ has radius of convergence unity, then as x increases to unity

$$\lim f(x) \text{ exists and} = \Sigma a_n,$$

if we know that Σa_n converges. The converse is not true, and if $\lim f(x)$ exists we say that Σa_n is summable A. Tauber in 1897 showed that a series summable A must converge if $na_n \to 0$, while in 1910 Littlewood proved the very much deeper result that this condition could be replaced by na_n bounded. Clearly such results, with their analogues for the Cesàro, Euler, Borel . . . methods of summation, due in so many cases to the Hardy-Littlewood collaboration, indicate in some sense the *capacity* of the summation process, and the literature of these Tauberian theorems is already so extensive that some guide to them is sure to be appreciated by those beginning research in this domain. Professor Karamata has published many papers on these topics, one of his earliest being a considerably simplified proof of Littlewood's theorem, and the present tract summarises a great deal of the relevant literature. T. A. A. B.

Détermination des fonctions entières par interpolation. By W. Gontcharoff. Pp. 48. 12 fr. 1937. Actualités scientifiques et industrielles, 465 ; exposés sur la théorie des fonctions, VII. (Hermann, Paris)

L'ultraconvergence dans les séries de Taylor. By G. Bourion. Pp. 46. 12 fr. 1937. Actualités scientifiques et industrielles, 472 ; exposés sur la théorie des fonctions, VIII. (Hermann, Paris)

These two tracts deal with subjects—interpolation and gap theorems—which are much to the fore at present. A comprehensive treatment of either, or, better still, of both, since they are related, would be very welcome ; but it is not possible to accomplish much in forty pages.

In this short space M. Gontcharoff manages to treat his subject with remarkable breadth. The accounts of the problem of interpolation in general and the work of Gelfond and others on the generalised Newton series are very welcome, as several of the papers are in Russian. The remainder of the tract deals with the generalised Abel series—the expansion of $f(z)$ in terms of $f(a_0)$, $f'(a_1), f''(a_2), \ldots$ —and calls attention to some unsolved problems.

M. Bourion's tract does not call for much notice as it covers ground readily accessible to the English reader in Dienes' *Taylor Series*, Chapter XI. The theorems of Jentzsch and Ostrowski naturally form the backbone and there is not much room for anything else. The presentation is interesting.

J. M. Whittaker.

Les Axiomes de la Mécanique Newtonienne. By Ch. Platrier. Pp. 58. 14 fr. 1936. Actualités scientifiques et industrielles, 427 ; exposés de mécanique newtonienne, I. (Hermann, Paris)

This account of the foundations of Newtonian mechanics is welcome at a time when, as is shown by recent elementary text-books, the fundamental

assumptions are explained in much more detail than formerly. (A well-known article in the *Gazette* on the " Teaching of Rational Mechanics " by the late Professor L. N. G. Filon is referred to in more than one such text-book, and is no doubt partly responsible for this change.) Professor Platrier's short book is part of the course given by him at l'École Polytechnique ; but it is, apart from important references to " La masse en cinématique et théorie des tenseurs du second ordre " in the same series (Actualités scientifiques et industrielles), self-contained, with an adequate account of what astronomy is needed for an understanding of the definitions. It is a useful book of reference for a first or second-year course at a university, although there are no illustrative examples in a field in which they can be made more than usually significant.

It is a little difficult to defend in one respect the order in which the ideas are presented. A mass is at first assigned to a body by supposing its weight compared on a balance with that of the selected body of unit mass. As a result in a table which compares English with C.G.S. units (" des unités anglaises assez complexes " is the inevitable description of the former), among English units of force appear the weights of the pennyweight, of the quarter, and even of the stone, while the poundal is not mentioned. Again later where a description of the stellar universe is given, the masses of the sun and planets are (effectively) stated at a time when they can be measured, so far as the given definitions are concerned, only by the common balance, and the *centre of gravity* of the solar system is stated to be a suitable origin for Newtonian axes. Inertia masses are not defined till late in the book, after the tables of units and the account of the solar system ; it would have been better in many ways to have started with these definitions. W. R. D.

Structure générale des nomogrammes et des systèmes nomographiques. By R. Lambert. Pp. 64. 15 fr. 1937. Actualités scientifiques et industrielles, 493 ; procédés généraux de calcul, I. (Hermann, Paris)

This tract is the first of a series on general methods of numerical calculation written by various authors, the whole series being edited by M. Maurice d'Ocagne. Starting from the simple case of single nomograms in one plane, M. Lambert develops the general theory in a systematic manner, considering in turn nomograms consisting of two or more planes and systems of nomograms. Consideration is confined to nomograms whose manipulation may be regarded as a practical proposition, that is to say, nomograms for which the answers may be read off directly and those whose method of operation involves a simple process of trial and error. The later chapters of the book are devoted to investigation of means by which the operation of systems of nomograms may be simplified.

The various special cases given as examples should provide useful suggestions for the practical construction of nomograms. B. M. B.

Briefwechsel Cantor-Dedekind. Edited by E. Noether and J. Cavaillès. Pp. 60. 20 fr. 1937. Actualités scientifiques et industrielles, 518. (Hermann, Paris)

This little book contains a number of letters of Cantor and Dedekind, published in German with a French preface. The letters of Cantor are those found among Dedekind's papers. The replies are incomplete. The letters make very good reading and help to bring back to life one of the most dramatic chapters in the history of mathematics.

The book is also of interest as a very valuable introduction to notions which are now of fundamental importance in Analysis. These notions are introduced tentatively in the letters, and then discussed in detail, criticised and modified,

if necessary. The student will find this much more instructive than the rigid form of presentation of a textbook. L. C. Y.

Educational, Psychological and Personality Tests of 1936. By OSCAR K. BUROS. Pp. 141. 75c. 1937. (School of Education, Rutgers University, New Brunswick, New Jersey)

This bibliography contains a useful list of all pencil and paper psychological tests published either in the United States or in the British Empire during 1936. Following upon similar lists for the years 1933, 1934, and 1935, it is planned as an annual publication, and should be indispensable as a reference for educational psychologists. The non-specialist English reader may find it difficult to examine the tests mentioned. The great majority of the tests are American, and hence difficult to obtain; the 27 entries included in the various mathematical sections, for instance, do not contain a single English title.

Of more immediate value to English readers, therefore, will be the list of 291 books and pamphlets on the subject of psychological testing, issued in 1936. Here some attempt has been made, by quoting from reviews in reputable journals, to provide a guide to this fantastic proliferation of reading-matter. This section is well conceived: where possible, the reviewer's name provides an extra index of the value of the quoted criticism and the extracts have been intelligently chosen. The general effect of these collocated extracts suggests the disinfected blurb of some uncommercially frank publisher.

M. B.

Projektive Geometrie. By K. DOEHLEMANN. New edition by H. Timerding. Pp. 131. RM. 1.62. 1937. Sammlung Göschen, 72. (Walter De Gruyter, Berlin)

The well-known *Sammlung Göschen* monographs need no introduction. This volume presents the principal theorems of plane and solid projective geometry in the compass of a hundred and thirty pages. It commences with a discussion of the elementary notions of plane geometry; cross-ratio, duality, harmonic ranges and involutions, and applies these to develop the geometry of the conic. In space it treats of quadrics and twisted cubics, of collineations and correlations, and of linear and tetrahedral line-complexes, giving in each case the salient theorems of the subject in as direct and concise a form as possible.

The lack of examples in the text will probably prevent the general acceptance of a work of this sort as a textbook in this country, but the volume (like others of the series) may be commended for giving in a brief compass a review of the main topics of projective geometry. J. A. T.

Versicherungsmathematik. I. Elemente der Versicherungsrechnung. By F. Böhm. 2nd edition. Pp. 151. RM. 1.62. 1937. Sammlung Göschen, 180. (Walter de Gruyter, Berlin)

This little work gives an elementary description of how to calculate the single and annual premiums in the more usual forms of Whole Life and Endowment assurances, and also shows how the reserve values are obtainable. It is written with clearness, and the detail suggests that the author is catering for the lowest forms of student, who require a little spoon-feeding.

The work contains 18 tables, six of them being the usual interest tables for the first 20 years with rates of interest from 3% to 5% (by $\frac{1}{2}$% intervals). The remaining tables give various results at $3\frac{1}{2}$% and are based on the 23 German Offices life table (of which the experience ended in the year 1875).

As the number in the Sammlung Göschen is 180 and this is the same as that of the earlier work by Professor Dr. Alfred Loewy, it is evidently designed

to take its place; this seems to us to be most unfortunate, and we would ask of the newer work, and continue asking like Gontan " Que diable alloit-il faire dans cette galère? "

The end and aim of the Sammlung Göschen is clearly printed on the inside front cover of each volume. The author has, however, chosen to ignore this almost entirely, for, with the exception that it should be easily understandable, it does not satisfy any of the conditions, such as, that it should be in the nature of a survey (*übersichtliche*) or that it should have regard to the most recent position (*des neuesten Standes*). Now Professor Loewy's book did fulfil these conditions, for he gave an account of the various mortality tables, British, American and German, and mentioned the newer 51 German Offices table which was in course of construction at the time when his second edition appeared (1910). Loewy also gave an account of select mortality tables, with the necessary modifications in the formulae. The fact is that Professor Loewy's book is suitable as an article for an encyclopaedia, and is of permanent value to the actuary or mathematician, while the new work can only find its place in the school class room.

One illustration of the simple nature of Dr. Böhm's work will suffice. On page 63 the author arrives at the result

$$nA_x = {}_nE_x \cdot A_{x+n}$$

(a result obvious to anyone who understands the meaning of the symbols) by six successive steps, one of which is the rather palsied step

$$\frac{D_{x+n} - d \,.\, N_{x+n}}{D_x} = \frac{D_{x+n}}{D_x} - d\frac{N_{x+n}}{D_x}.$$

Is there room for this kind of thing in this valuable Sammlung? W. S.

A School Algebra. I. II. By R. M. Carey. Pp. viii, 288. With answers, 4s.; without answers, 3s. 6d. 1936. (Longmans, Green)

The most useful features of this algebra book would appear to be the unusually large sets of examples and the reasonably up-to-date treatment of the subject. Mr. Carey claims that he has tried to do justice to the formula and to the graph, and he has tried to rearrange the subject-matter so that progress shall be rapid. For instance, speed problems and four-term factors have been postponed, and fractions have been taken in three stages.

The book begins with graphs (this is unusual), continues with equations and brackets, and reaches formulae rather later than might have been expected. For instance, in discussing manipulation of algebraic expressions, there are considerable possibilities in using formulae for areas and volumes. Incidentally one deficiency in the book is the paucity of diagrams. Transformation of formulae is discussed at an early stage. The bookwork of negative numbers is purposely cut down, because the teacher will have to supply it, but the bookwork of other sections is occasionally rather unwisely uneven. For instance, the opening chapter on graphs is formidable for a young beginner, as some of the examples contain awkward numbers: the construction of formulae on page 77 is uninspiring, and it seems a pity that problems on the quadratic should be left until the pupil has been taken through solution by factors, by graph, and by completing the square. In fact it appears that after a promising start, Mr. Carey has lost sight of the formula and the graph. Certainly the formula seems to fade out when manipulation of quadratic equations appears on the scene, and hardly any graphs are illustrated save those of quadratic functions or straight lines. There are a few signs of haste: *e.g.* the phrases " place on a common denominator " and " many ideas such as powers, fractions and L.C.M. are familiar in arithmetic " seem awkward, most of the

graph on page 83 is wasted, and Question 7 on page 208 is distinctly puzzling if not meaningless. One feels that if Mr. Carey had compiled his material more from real sources and had made the formula and the graph his real foundation stones, he would have written a more interesting book. However, if teachers can supply this material for themselves, they have at their disposal in this book a great collection of formal material from which to select their examples.

C. T. DALTRY.

Arithmetic of Daily Life. I. By H. WEBB and J. C. HALL. Pp. 96. 1s. 3d. 1937. (Cambridge)

It is not easy to review this book properly. The authors say nothing about themselves or their work. They do not say for whom the book has been written, on what plan it has been arranged, or how many succeeding books will complete the whole course. There is no preface. I can only guess that the book is written for senior school pupils or for less able pupils in secondary schools, and that it covers a year's work. But the arrangement baffles me. The first part is a revision course in arithmetic, but the second and third parts are geometry and algebra. The pupil appears to be expected to know the rudiments of arithmetic, and the course is mostly exercises and tests. The arithmetic of daily life appears in sections on bus conductors' way bills, part-time employment, behind the counter, at the post office, etc. These are really interesting, but much of the other material is humdrum. The sections on geometry and algebra are unimpressive and much below the best work in these subjects in modern text-books. The illustrations by Bert Thomas are enlivening. They are something new, and suggest interesting developments in illustrated mathematical books. I trust that the omission of prefaces is *not* a new idea and that it will not be copied. The printing is excellent.

C. T. DALTRY.

Sane Arithmetic for Seniors. I. By C. WARRELL. Pp. 64. Paper, 10d.; limp cloth, 1s. 3d. 1937. (Harrap)

This is a straightforward book for senior school pupils. It covers the arithmetic of number, everyday units (money, length, weight), diagrams (using simple fractions) by selected topics. Each page contains exercises on a topic, such as milk for schools, planning a garden, buying a bicycle. The book does not include manipulation of fractions or decimals, but it contains a very attractive presentation of the work that might precede these in every arithmetic course. This book seems much more real and useful than most elementary arithmetic books. It is well printed and is worth inspection.

C. T. DALTRY.

Scripta Mathematica Forum Lectures. Addresses by C. J. KEYSER, D. E. SMITH, E. KASNER and W. RAUTENSTRAUCH. Pp. 94. $1; postage 10 cents. 1937. The *Scripta Mathematica* Library, 3. (Yeshiva College, New York)

The first essay, entitled "Mind, the Maker" is a presentation by C. J. Keyser of the world-theory of W. B. Smith. It does not make easy reading, even to one who knows his Carlyle fairly well, and the mathematician will probably feel that these are haunts not meet for him.

Professor D. E. Smith in "The Story of Mathematics" endeavours to describe the rise and fall of mathematics in various countries and ages; naturally 19 pages suffice for only the barest outline. Perhaps this essay will

stimulate the production of thorough investigation of particular periods in relation to political, theological and economic factors.

Professor Kasner on " New Names in Mathematics " comments facetiously on the complex nature of a " simple " curve and of a " simple " group, and instances the utility of easy words for hard ideas, such as " cycle " for the oriented circle. He also suggests some new names, including " googol " for unity followed by one hundred zeros and " googolplex " for unity followed by a googol of zeros ; these are odd names, but the mathematician who knows why a " yorker " is so named will not raise serious objections.

" Science in Relation to Social Growth and Economic Development ", by Professor Rautenstrauch, is an essay by an engineer who believes that since private enterprise has failed to secure the economic well-being of the people as a whole, the time has come for " mankind to organise after the pattern of the University those processes by which it obtains clothing, food and shelter, and the material things of life ". His ideal is an economic nationalism, a kind of diluted Fascism, producing prosperous and healthy citizens, each helping the creative activities of the State, an engineer's Utopia. T. A. A. B.

Analisi matematica algebrica ed infinitesimale. By B. LEVI. Pp. vii, 541. L. 80. 1937. (Zanichelli, Bologna)

Italian text-books on analysis are probably not so well known in this country as the numerous excellent Italian treatises on geometry. Professor Levi, whose name is familiar in connection with the Lebesgue integral, provides a straightforward course containing roughly the algebra and calculus of the non-specialist part of the Tripos. The theory is set out clearly, and is illustrated by numerous comments and examples in small type ; there are no examples for the reader. Historical notes are frequently given, and most of the mathematicians mentioned are provided with their dates ; unfortunately the only dates the reviewer could check without using a reference book, those of Newton, are incorrect.

The first four chapters deal with polynomials, matrices, determinants, complex numbers, vectors * and quaternions. In the fifth chapter, on continuity, an introduction is given to the theory of aggregates, and the General Principle of Convergence is proved to be a necessary condition for the existence of a limit ; but so far nothing has been said about irrational numbers, and so the sufficiency of this condition in the field of real numbers cannot be proved. The difficulty is met by a form of Dedekind's postulate, whereby we postulate the existence of a limit if the General Principle of Convergence holds. Russell's opinion of the logical advantages of " postulating " when we cannot " prove " is too well known to need quoting ; whether the teaching advantages are open to the same objection is another question, but it seems likely that a student acute enough to see the difficulty would be wary of receiving stolen goods and robust enough to profit by an account of the Frege-Russell methods.

Following this there is a chapter on limits and then one on power-series ; the theory is centred round the expression

$$R = 1/\overline{\lim}\,|\,a_n\,|^{1/n}$$

for the radius of convergence of a power-series, and the proof that the exponential series converges in the whole complex plane is thus a little complicated. The exponential function is introduced here as the sum of the power-series ; there is no reason given for considering this series rather than

* It is rash for anyone who likes a quiet life to criticise a vector notation, but the cross as the sign of a *scalar* product seems an unfortunate choice.

some other. but the book is not intended for the novice but for the maturer student who will know quite well why this series is important.

Chapters VIII and IX deal with derivatives. In Chapter VIII we are concerned with fields in which

$$\{f(x)-f(x_0)\}/(x-x_0)$$

tends to a limit as $x \to x_0$, in IX with the special properties of this limit when $f(x)$ is a real function of the real variable x.

Chapter XI deals with the fundamental theorem of algebra and with other properties of equations. Then there are chapters on indefinite integration and on the definite integral with some references to the Lebesgue and Stieltjes integrals. The section on Stieltjes integrals might profitably have been expanded, since the student reading a book of this type on analysis ought to have the chance of learning more than is here given about these useful integrals.

Chapter XIV is a mixed bag of applications: Newton's iteration for the roots of an equation, the Fourier-Budan theorem, Wallis' product, Stirling's approximation for $n!$, Fourier series and integrals. The final chapters concern differential equations, functions of several variables, and geometrical applications.

The general effect on the English reader is of traversing well-known paths in an unusual order, with the author always able to show us something new or, more frequently, something known but seen here from a new point of view. There is a wealth of material in the 500 pages, and, in general, an ease of exposition which makes the volume an excellent and by no means superfluous addition to the many treatises on analysis.

It is a pleasure to record that in addition to a detailed table of contents there is a reasonably complete index and a list of names quoted. T. A. A. B.

The teaching of mathematics. A bibliography. By M. Black. Pp. 14. 6d. N.d. (Christophers)

" This list has been compiled in the hope that it may be useful to those who wish for better acquaintance with the enormous and increasing literature of books on the teaching of mathematics ". It is arranged under six main headings: popular introductions; philosophy of mathematics; history; methods of teaching; supplementary material; further bibliographies. The compiler's brief comments on individual items are concise, to the point and almost always accurate. Very few of the books mentioned appear to the reviewer to be unworthy of their place; naturally the number of good books not mentioned is large, but that must happen with any small, highly selective list. The young teacher will find this pamphlet a useful guide T. A. A. B.

EXCHANGE WANTED

The Secretary has particulars from a member in South Africa who wishes to arrange an exchange (for 1939) with a teacher of mathematics in England. Any member who is interested is asked to apply to Mr. Parsons without delay.

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